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# Para-Hermitian Geometry, Dualities and Generalized Flux Backgrounds

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## Abstract

We survey physical models which capture the main concepts of double field theory on para-Hermitian manifolds. We show that the geometric theory of Lagrangian and Hamiltonian dynamical systems is an instance of para-Kähler geometry which extends to a natural example of a Born geometry. The corresponding phase space geometry belongs to the family of natural almost para-Kähler structures which we construct explicitly as deformations of the canonical para-Kähler structure by non-linear connections. We extend this framework to a class of non-Lagrangian dynamical systems which naturally encodes the notion of fluxes in para-Hermitian geometry. In this case we describe the emergence of fluxes in terms of weak integrability defined by the D-bracket, and we extend the construction to arbitrary cotangent bundles where we reproduce the standard generalized fluxes of double field theory. We also describe the para-Hermitian geometry of Drinfel'd doubles, which gives an explicit illustration of the interplay between fluxes, D-brackets and different polarizations. The left-invariant para-Hermitian structure on a Drinfel'd double in a Manin triple polarization descends to a doubled twisted torus, which we use to illustrate how changes of polarizations give rise to different fluxes and string backgrounds in para-Hermitian geometry.

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# 1 Introduction and Overview

Para-Hermitian geometry has acquired renewed interest in recent years because of its relevance to flux compactifications of string theory, which is inspired by its connections to generalized geometry and double field theory. Before describing the new impetus that the present paper provides to this endeavour, let us set the stage by briefly recalling the connections between generalized geometry, double field theory and para-Hermitian geometry.

**Generalized Geometry.** Generalized geometry [1, 2] is a powerful mathematical framework in which a unified description of vector fields and 1-forms is achieved. It is a framework in which dualities, which typically emerge in physical theories with extra dimensions such as string theory, can be naturally studied. It is particularly relevant for the description of T-duality of a string background  $(g, B)$  on a  $d$ -dimensional target space, where  $g$  is the spacetime metric (which we take to be of Euclidean signature for definiteness) and  $B$  is the Kalb-Ramond field. In the low-energy limit, string theory is described by supergravity and the bosonic part of the effective action is given by

$$S_{\text{SUGRA}} = \int d^d x \sqrt{g} e^{-2\phi} \left( \text{Ric}(g) + 4 \partial_i \phi \partial^i \phi - \frac{1}{12} H_{ijk} H^{ijk} \right), \quad (1.1)$$

where  $H = dB$  is the NS-NS  $H$ -flux,  $\phi(x)$  is the string dilaton field and  $\text{Ric}(g)$  is the Ricci scalar of the metric  $g$ . The field equations resulting from this action impose vanishing  $\beta$ -functions of the background  $(g, B)$  and the dilaton  $\phi$ , ensuring conformal invariance of the string theory at 1-loop order in worldsheet perturbation theory.

The action (1.1) has a non-manifest  $O(d, d)$ -symmetry. In generalized geometry this appears in a generalized tangent bundle over the target space which has the structure of a Courant algebroid with fiber metric  $\eta$  of signature  $(d, d)$  and the Courant bracket of its sections which are generalized vector fields. The generalized metric  $\mathcal{H}$  on a generalized tangent bundle encodes all information about a given background. For compactifications on torus fibrations, the  $O(d, d)$ -symmetry includes T-duality which relates the backgrounds of two non-linear worldsheet sigma-models with different target spaces. In this case T-duality transformations of the generalized metric appear as isomorphisms between the generalized tangent bundles of principal torus bundles endowed with a closed 3-form [3], viewed as (twisted) Courant algebroids. In this way the symmetries of string theory have a natural interpretation in terms of structures emerging from generalized geometry.

**Double Field Theory.** Dualities in theories with extra dimensions typically indicate the presence of hidden symmetries of the theory, and it is natural to search for an extended theory where these become manifest symmetries. T-duality may be a key to understanding the structures characterizing the spacetime of a theory with extra dimensions, and in this case the extended theory should be a manifestly  $O(d, d)$ -invariant theory. The extended theory is then formulated on a doubled target space with generalized coordinates  $x^I = (x^i, \tilde{x}_i)$ , where the spacetime coordinates  $x^i$  and their duals  $\tilde{x}_i$  naturally emerge from string theory where they are fields appearing in dual actions related by  $O(d, d)$ -transformations. In this way T-dualities may play an even bigger role as part of the diffeomorphisms of such a space [4, 5], and this is the starting point in formulating a theory which is manifestly invariant under T-duality transformations.

It is clear that generalized geometry is not the framework in which a doubled target space can

be implemented, as it only doubles the fibres of the tangent bundle over the original spacetime. Only a theory defined on a doubled geometry, as proposed in [4, 5], is possible and a T-duality invariant theory called double field theory has been suggested for this purpose [6–8]. It is related to the low-energy limit of bosonic string theory, since it gives an effective action which is T-duality invariant only if it is formulated on a doubled space [9, 10] and is constructed in the following way. The doubled space has two metrics, the constant metric  $\eta$  of signature  $(d, d)$  which raises and lowers indices in the following, and the Riemannian metric  $\mathcal{H}$  which incorporates the dynamical fields  $(g, B)$ . Starting from the low-energy effective action (1.1), the action of double field theory can be written using the  $O(d, d)$ -tensor  $\mathcal{H}^{-1}$  as

$$S_{\text{DFT}} = \int d^d x d^d \tilde{x} e^{-2\phi} \left( \frac{1}{8} \mathcal{H}^{IJ} \partial_I \mathcal{H}^{KL} \partial_J \mathcal{H}_{KL} - \frac{1}{2} \mathcal{H}^{IJ} \partial_J \mathcal{H}^{KL} \partial_L \mathcal{H}_{IK} \right. \\ \left. - 2 \partial_I \phi \partial_J \mathcal{H}^{IJ} + 4 \mathcal{H}^{IJ} \partial_I \phi \partial_J \phi \right), \quad (1.2)$$

which is manifestly  $O(d, d)$ -invariant [10].

The action (1.2) reduces to (1.1) upon imposing a constraint which halves the number of coordinates; this is called the *section condition*. In particular, we can consider the level matching condition  $L_0 - \bar{L}_0 = 0$  of the worldsheet theory, which in the doubled formalism becomes  $\partial_M \partial^M = 0$  on any field and any parameter; this is called the strong constraint and one of its solutions is obtained by taking all fields to be independent of the dual coordinates  $\tilde{x}_i$ , in which case we recover (1.1).

The action (1.2) is also invariant under diffeomorphisms which are generated by doubled vector fields and a suitable notion of generalized Lie derivative. The generalized Lie derivative of the metric  $\eta$  vanishes, which implies that the constraint for  $\mathcal{H}$  to be an  $O(d, d)$ -tensor is compatible with diffeomorphisms, while the generalized Lie derivative of the identity  $\mathcal{H} \eta \mathcal{H} = \eta^{-1}$  gives the compatibility between the  $O(d, d)$  and gauge symmetries. The  $O(d, d)$ -covariant extension of the Courant bracket for doubled vector fields is called the *C-bracket* and is given by

$$([\xi_1, \xi_2]^C)^I = \xi_{[1}^J \partial_J \xi_{2]}^I - \frac{1}{2} \xi_{[1}^J \partial^I \xi_{2]}{}_J. \quad (1.3)$$

The C-bracket governs the algebra of generalized Lie derivatives. Although this theory still lacks a global formulation, it suggests a close relationship with generalized geometry, at least locally, since the Courant algebroid structure of the generalized tangent bundle is recovered upon imposing the section condition.

Having the algebra of gauge transformations at hand, one can now write the gauge-invariant action in terms of  $\mathcal{H}$ . This action is manifestly  $O(d, d)$ -invariant and is expressed in an Einstein-Hilbert type form [9, 10], where the generalized scalar curvature is defined as the field equation for the dilaton following from the action (1.2). The equations of motion combine the field equations for the metric  $g$  and the  $B$ -field in an  $O(d, d)$ -covariant form, which has the form of the Einstein equation extended to the doubled formalism. In this sense double field theory can be regarded as a low-energy effective field theory of string generalized geometry. For more detailed reviews of double field theory, see [11–13]. We stress that this is only a local formulation of the theory, and the problem is to find a suitable geometry in which all these structures can be defined globally.

**Para-Hermitian Geometry.** The discussion thus far stresses the need of a mathematical framework in which a global formulation of doubled geometry is possible, while at the same recovering

a precise relation with generalized geometry. A promising approach is achieved by *para-Hermitian geometry*, which was first proposed in the context of a global formulation of double field theory in [14, 15], and developed further by [16, 17] to provide a global formulation of its kinematics. A unified approach was presented in [18, 19] where a further generalization of the notion of Courant algebroid is given in order to encode the desired features of a doubled target space. However, the first appearance of para-Hermitian geometry in the description of T-duality can be traced back to the work of [4], where a T-fold is described in terms of doubled torus bundles in which para-Hermitian structures are defined on the fibers. We review this approach in some precise detail in Section 2 below, and present here a brief overview of the relevant aspects while glossing over many technicalities. For a more complete survey of the mathematical aspects of para-Hermitian geometry, see [20].<sup>1</sup>

Para-Hermitian geometry is, roughly speaking, formulated as a real version of the more familiar concepts from complex, Hermitian and Kähler geometry. A *para-complex structure* on a vector bundle  $E \rightarrow M$  of even rank  $2d$  over a manifold  $M$  is a bundle endomorphism  $K \in \text{End}(E)$  such that  $K^2 = \mathbb{1}$  and the  $\pm 1$ -eigenbundles of  $K$  have the same rank  $d$ . A symmetric non-degenerate pairing  $\eta$  of sections of  $E$  is called para-Hermitian if  $\eta(K(X), K(Y)) = -\eta(X, Y)$  for all  $X, Y \in \Gamma(E)$ , and the pair  $(K, \eta)$  is then called a *para-Hermitian structure* on  $E$ . A *para-Hermitian manifold* is a manifold  $M$  whose tangent bundle carries a para-Hermitian structure; the compatible (almost) para-complex structure  $K$  and Lorentzian metric  $\eta$  naturally give rise to a fundamental 2-form  $\omega$  on  $M$ , and if  $\omega$  is closed then  $(M, K, \eta)$  is called a *para-Kähler manifold*.

In the framework for double field theory on para-Hermitian manifolds, the original spacetime  $F$  is regarded as a submanifold of the doubled space  $M$  when one of the eigenbundles of the para-complex structure is integrable. The generalized geometry perspective is recovered as the tangent bundle  $TM$ , whose metric  $\eta$  defines a bundle isomorphism from  $TM$  to the generalized tangent bundle  $TF \oplus T^*F$  on the integral foliation  $F$  of the integrable distribution. Para-Hermitian connections are defined as connections on  $TM$  which preserve both  $K$  and  $\eta$ , and in particular the parallel transport of sections of the eigenbundles; the Levi-Civita connection of the  $O(d, d)$ -metric  $\eta$  is a para-Hermitian connection only on para-Kähler manifolds. One then defines the D-bracket of vector fields on  $M$  in terms of a canonical para-Hermitian connection which derives from the Levi-Civita connection, and its skew symmetrization gives the C-bracket which has the local expression (1.3) when the para-Hermitian structure is flat. The bundle isomorphism above then sends the C-bracket on  $TM$  to the Courant bracket of the exact Courant algebroid on  $TF \oplus T^*F$  [16, 17, 19]. This result clarifies the differences between generalized geometry and doubled geometry, and states a precise relation between them.

Different dual spacetimes are obtained as different para-complex structures, i.e. *polarizations*, on the tangent bundle  $TM$ . In particular, suitable spacetimes are Lagrangian submanifolds of the doubled space  $M$  which, when endowed with a generalized metric  $\mathcal{H}$  encoding the data of the background fields, gives rise to the notion of a *Born geometry* [24, 25]. Furthermore, a first appearance of geometric and non-geometric fluxes appears in this formalism through deformations of para-Hermitian structures by  $B$ -transformations [19], which are endomorphisms of  $TM$  that preserve the metric  $\eta$  but twist the almost para-complex structure, the fundamental 2-form, and the D-bracket. These constructions thus far fulfill the requirements of a global formulation for the kinematics of double field theory, whereas a global description of its dynamics is still lacking.

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<sup>1</sup>An alternative mathematical approach to double field theory based on graded symplectic geometry is discussed in [21–23].

**Overview of Results and Outline.** This paper is concerned with the description of physical spacetimes, their backgrounds and their duality transformations in the setting of para-Hermitian geometry; we set the stage for this in Section 2 by first briefly reviewing aspects of para-Hermitian geometry, following [16, 17, 19] for the most part, and elucidating the discussion to describe global aspects of polarizations and T-duality in this framework. Our aim is to focus on some explicit classes of examples in which (almost) para-Hermitian structures naturally arise, and from them extract features of duality transformations and the fluxes which characterize each polarization. Our examples are mostly known already in the literature, though perhaps less widely in the context of para-Hermitian geometry, which is the geometric impetus that we emphasise through the physics of these examples. Through these explicit settings we can then extract general properties of dualities and how generalized fluxes appear on para-Hermitian manifolds, and understand better their global physical features.

As a first class of examples, we show that the Lagrangian (and Hamiltonian) description of classical dynamical systems leads to a natural para-Kähler structure on the tangent bundle  $M = T\mathcal{Q}$  of the underlying configuration space  $\mathcal{Q}$ . Together with the dynamical almost Hermitian structure determined by the Lagrangian function, the para-Kähler structure gives a natural instance of Born geometry. The Finsler geometry of regular Lagrangian dynamical systems is also connected to double field theory and generalized geometry in [15, 26], whereas in this paper we emphasize their geometric aspects in terms of para-Kähler structures. We describe the Legendre transform to the cotangent bundle  $T^*\mathcal{Q}$  as a member of the most general family of para-Kähler structures on phase space, which are obtained as deformations of the canonical para-Kähler structure by symmetric  $(0, 2)$ -tensor fields  $C$  on  $T^*\mathcal{Q}$  which are coefficients of torsion-free non-linear connections on the tangent bundle  $T(T^*\mathcal{Q})$ . This generalizes the constructions of [15, 26, 27] in which  $C$  is taken to be a natural lift to  $T^*\mathcal{Q}$  of the Christoffel symbols of the Levi-Civita connection of a Riemannian metric  $g$  on the configuration manifold  $\mathcal{Q}$ . The para-Kähler geometry underlying classical dynamical systems in both the Lagrangian and Hamiltonian formalisms is described in Section 3.

We then turn to the extension of this construction to cases where a Lagrangian (and Hamiltonian) is not globally defined. We consider a prototypical class of examples which includes, as specific cases, the motion of charged particles in magnetic fields generated by distributions of magnetic monopoles and the motion of closed strings in locally non-geometric  $R$ -flux backgrounds [28, 29]. We provide an interpretation of these dynamical systems in terms of deformations of para-Kähler structures in order to explain how fluxes emerge in this case, which also lends a novel perspective to their inherent nonassociativity. In particular, we can recover the almost symplectic 2-form  $\omega_B$  defining the twisted Poisson brackets of the dynamical system as the fundamental 2-form  $\omega_B = \eta_0 K_B = \eta_0 K_0 + 2\eta_0 B$  of the almost para-Hermitian structure  $(K_B, \eta_0)$  on  $T(T^*\mathcal{Q})$ , where  $K_B$  determines the splitting  $T(T^*\mathcal{Q}) = L_v \oplus L_h^B$  with<sup>2</sup>  $L_v = \text{Span}_{C^\infty(T^*\mathcal{Q})}\{Q^i\}$  and  $L_h^B = \text{Span}_{C^\infty(T^*\mathcal{Q})}\{D_i = P_i + B_{ij}Q^j\}$ , and  $\eta_0$  is a flat metric of Lorentzian signature; it follows that  $L_v$  is an integrable distribution while  $L_h^B$  is not. The almost para-complex structure  $K_B$  can be regarded as a  $B$ -transformation of the para-complex structure  $K_0$  with integrable eigenbundles  $L_v$  and  $L_h^0 = \text{Span}_{C^\infty(T^*\mathcal{Q})}\{P_i\}$ , which illustrates the general feature that a  $B$ -transformation does not preserve integrability.

If the endomorphism  $B$  depends only on the configuration space coordinates  $q$ , then we show

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<sup>2</sup>Notation:  $Q^i = \frac{\partial}{\partial p_i}$  and  $P_i = \frac{\partial}{\partial q^i}$ , where  $(q^i, p_i)$  are local Darboux coordinates on  $T^*\mathcal{Q}$ . We write  $\partial_i f = P_i(f)$  and  $\tilde{\partial}^i f = Q^i(f)$  for any function  $f \in C^\infty(T^*\mathcal{Q})$ .

that the only non-vanishing D-bracket with respect to  $K_B$  is

$$\llbracket D_i, D_j \rrbracket_B^{\mathbb{D}} = \frac{3}{2} \partial_{[i} B_{jk]} Q^k .$$

This bracket is thus related to the  $H$ -flux and is an element of  $\Gamma(L_{\mathbf{v}})$  (the precise relation can be found in [19] and Section 2.3 below), which means that the  $H$ -flux obstructs the integrability of the  $B$ -transformed distribution and a foliated manifold with local momentum coordinates  $p$  does not exist. Assuming more generally that  $B$  may also depend on the fiber coordinates  $p$ , we show that the D-bracket becomes

$$\llbracket D_i, D_j \rrbracket_B^{\mathbb{D}} = \frac{3}{2} ((\partial_{[i} B_{jk]} + B_{[im} \tilde{\partial}^m B_{jk]}) Q^k + \tilde{\partial}^k B_{ij} D_k) ,$$

where now integrability is obstructed by the covariant  $H$ -flux  $\mathcal{H}_{ijk} = \partial_{[i} B_{jk]} + B_{[im} \tilde{\partial}^m B_{jk]}$ , which is the  $\Gamma(L_{\mathbf{v}})$ -component of the D-bracket.

This construction is then extended to give the full set of geometric and non-geometric fluxes of (local) double field theory, extending the considerations of [19]. In particular, we recast Born reciprocity in terms of these deformations and extend the discussion to describe generalized fluxes as deformations of the para-Kähler geometry of the cotangent bundle  $T^*\mathcal{Q}$  involving both  $B$ - and  $\beta$ -transformations. This gives an alternative perspective in the setting of para-Hermitian geometry to closed string noncommutativity and nonassociativity in non-geometric flux backgrounds (see [30] for a review). The appearance of nonassociativity in this way from changes of polarizations and flux deformations of the canonical para-Kähler structure of closed string zero modes was also discussed in [31], and our complimentary detailed construction further elucidates their meaning in terms of violations of weak integrability. The role of polarizations of the para-Hermitian geometry on the cotangent bundle  $T^*\mathcal{Q}$  was also emphasised by [32, 33] in relating phase space and spacetime nonassociativity. The description of fluxes in non-Lagrangian dynamical systems and in the dual  $R$ -flux model is the subject of Section 4.

We finally consider the broad classes of non-trivial doubled manifolds provided by Drinfel'd doubles and doubled twisted tori, in which para-Kähler structures can be naturally defined, and describe their different polarizations as duality transformations of these structures along with the related fluxes. In Section 5 we recall the well-known para-Hermitian geometry of Drinfel'd doubles and compute the corresponding D-brackets to illustrate how fluxes arise. Drinfel'd doubles extend the case of cotangent bundles and provide non-abelian generalizations of the standard flat para-Kähler doubled tori  $T^{2d}$ , while their different Manin triple polarizations generalize the different ways of embedding  $T^d \subset T^{2d}$  which are related by the action of the T-duality group  $\mathrm{O}(d, d; \mathbb{Z})$ . The para-Hermitian geometry of Drinfel'd doubles treats non-abelian T-duality using  $\mathrm{O}(d, d)$ -type structures as in [34], whereas more general polarizations than those provided by Manin triples define a modified non-abelian T-duality group and enable the introduction of generalized fluxes. In Section 6 we apply the framework of para-Hermitian geometry to describe doubled twisted tori, which comprise a class of well-known examples of global non-trivial doubled geometry, and analyse their different polarizations together with their backgrounds. This gives a more intrinsic perspective on the T-duality transformations relating almost para-Hermitian structures on the doubled twisted torus which are discussed in [35].



## 2 Para-Hermitian Geometry, Polarizations and T-Duality

$O(d, d)$ -transformations of supergravity can be described within the mathematical framework of generalized geometry. They include T-dualities between string backgrounds, which live in the discrete subgroup  $O(d, d; \mathbb{Z})$ , and hence they must be included in the geometric structure of an  $O(d, d)$ -invariant theory. The goal of double field theory is the description of an effective theory which manifestly possesses this invariance. Doubled geometry is the proposed mathematical framework in which such a theory should be formulated, but it currently lacks a global description, despite bearing similarities with generalized geometry [4, 5]. As first proposed in [14, 15], para-Hermitian geometry may provide the natural framework in which a global formulation of double field theory can be achieved, since it has an intimate connection with generalized geometry; in this setting “doubled spacetime” is synonymous with “para-Hermitian manifold”. In this section we will briefly review aspects of para-Hermitian geometry that we need in this paper, following [16, 17, 19] where a description of the foundations of double field theory is provided, as well as a first instance of how to introduce fluxes into this framework.

Para-Hermitian geometry first played a central role in the discussion of T-duality in [4], where para-Hermitian structures are defined on the fibers of a doubled torus bundle  $M \rightarrow W$  to provide the first geometric definition of a T-fold; the various polarizations of the  $T^{2d}$  fibers give different T-dual backgrounds. The almost product structures associated to each T-dual polarization are related by  $O(d, d)$ -transformations. In the setting of para-Hermitian geometry, we will clarify below in which sense a physical spacetime can only be recovered locally for a globally non-geometric background by relating it to the integrability of the distribution associated to it as a choice of the almost product structure in a general global formulation. Locally non-geometric backgrounds on the other hand, where not even a local spacetime description is possible, are particular polarizations of a doubled twisted torus in which the base space  $W \times \tilde{W}$  is also doubled [36] and will be characterized globally in the following in terms of non-integrable distributions. We return to the specific examples of doubled twisted tori using this general formalism in Section 6.

### 2.1 Para-Hermitian Manifolds

Throughout this paper all manifolds are assumed to be smooth.

**Definition 2.1.** An *almost product structure* on a manifold  $M$  is a  $(1, 1)$ -tensor field  $K \in \text{End}(TM)$  such that  $K^2 = \mathbb{1}$ . The pair  $(M, K)$  is an *almost product manifold*.

Fixing a coordinate chart  $(U, \phi)$  on  $M$  with local coordinates  $x^i$ , an almost product structure can be written as  $K = K_i^j dx^i \otimes \frac{\partial}{\partial x^j}$  on  $U$  with  $K_i^m K_m^j = \delta_i^j$ . In this definition the analogy with almost *complex* manifolds is clear, i.e. even-dimensional manifolds endowed with a  $(1, 1)$ -tensor field  $J$  such that  $J^2 = -\mathbb{1}$ . This analogy is a useful guide to understanding the structures introduced in the following and will be often recalled by the terminology we adopt.

**Definition 2.2.** An *almost para-complex manifold* is an almost product manifold  $(M, K)$  with  $M$  of even dimension such that the two eigenbundles  $L_+$  and  $L_-$  associated, respectively, with the eigenvalues  $+1$  and  $-1$  of  $K$  have the same rank. A splitting of the tangent bundle  $TM = L_+ \oplus L_-$  of a manifold  $M$  into the Whitney sum of two sub-bundles  $L_+$  and  $L_-$  of the same fiber dimension is an *almost para-complex structure* on  $M$ .

Recall that a  $\mathbf{G}$ -structure on a  $2d$ -dimensional manifold  $M$ , for a subgroup  $\mathbf{G} \subset \mathrm{GL}(2d, \mathbb{R})$ , is a  $\mathbf{G}$ -sub-bundle of the frame bundle  $FM$ , i.e. a reduction on the frame bundle of the structure group  $\mathrm{GL}(2d, \mathbb{R})$  to  $\mathbf{G}$ . Using this notion, we can rephrase the definition of almost para-complex structure by saying that it is a  $\mathbf{G}$ -structure on  $M$  with structure group  $\mathbf{G} = \mathrm{GL}(d, \mathbb{R}) \times \mathrm{GL}(d, \mathbb{R})$ .

Using the almost product structure, we can define two projection operators

$$\begin{aligned}\Pi_+ &= \frac{1}{2}(\mathbb{1} + K) : \Gamma(TM) \longrightarrow \Gamma(L_+) , \\ \Pi_- &= \frac{1}{2}(\mathbb{1} - K) : \Gamma(TM) \longrightarrow \Gamma(L_-) .\end{aligned}$$

Then we are naturally led to study the integrability of the sub-bundles  $L_+$  and  $L_-$ .

**Definition 2.3.** An almost product structure  $K$  is (*Frobenius*) *integrable* if its sub-bundles  $L_+$  and  $L_-$  are both integrable, i.e.  $[\Gamma(L_+), \Gamma(L_+)] \subseteq \Gamma(L_+)$  and  $[\Gamma(L_-), \Gamma(L_-)] \subseteq \Gamma(L_-)$ . An integrable almost product structure is a *product structure*. A *para-complex structure* is an integrable almost para-complex structure, i.e. a product structure with  $\mathrm{rank}(L_+) = \mathrm{rank}(L_-)$ .

By Frobenius' Theorem, this means that the manifold  $M$  admits two foliations  $\mathcal{F}_+$  and  $\mathcal{F}_-$ , such that  $L_+ = T\mathcal{F}_+$  and  $L_- = T\mathcal{F}_-$ .

Another way to characterize the integrability of an almost product structure is through the definition of the Nijenhuis tensor field, continuing the analogy with almost complex structures.

**Definition 2.4.** The *Nijenhuis tensor field* of an almost product structure  $K$  is the map  $N_K : \Gamma(TM) \times \Gamma(TM) \rightarrow \Gamma(TM)$  given by

$$N_K(X, Y) = [X, Y] + [K(X), K(Y)] - K([K(X), Y] + [X, K(Y)]) ,$$

for all vector fields  $X, Y \in \Gamma(TM)$ .

**Theorem 2.5.** An almost product structure  $K$  on a manifold  $M$  is integrable if and only if  $N_K(X, Y) = 0$  for all  $X, Y \in \Gamma(TM)$ .

Using the projection tensors  $\Pi_+$  and  $\Pi_-$ , together with  $K = \Pi_+ - \Pi_-$ , we can decompose the Nijenhuis tensor as

$$N_K(X, Y) = N_{\Pi_+}(X, Y) + N_{\Pi_-}(X, Y) ,$$

where

$$N_{\Pi_+}(X, Y) = \Pi_-([\Pi_+(X), \Pi_+(Y)]) \quad \text{and} \quad N_{\Pi_-}(X, Y) = \Pi_+([\Pi_-(X), \Pi_-(Y)]) . \quad (2.6)$$

From (2.6) we evidently have  $N_{\Pi_+}(X, Y) \in \Gamma(L_-)$  and  $N_{\Pi_-}(X, Y) \in \Gamma(L_+)$ . Hence the two components of the Nijenhuis tensor obstruct the closure of the Lie bracket of vector fields restricted to  $L_+$  and  $L_-$ , respectively. In particular,  $N_{\Pi_+}$  and  $N_{\Pi_-}$  are independent of each other. For instance, we may have  $N_{\Pi_+}(X, Y) = 0$  and  $N_{\Pi_-}(X, Y) \neq 0$ , so that the almost para-complex structure is only partially integrable ( $N_K(X, Y)$  is still non-vanishing) and it admits only one foliation  $\mathcal{F}_+$  such that  $L_+ = T\mathcal{F}_+$ . In this case we call  $(M, K)$  an  $L_+$ -*para-complex manifold*, i.e. there is a splitting of the tangent bundle  $TM$  into two distributions with the same rank such that only the eigenbundle associated to the eigenvalue  $+1$  is integrable. Exchanging the roles of  $N_{\Pi_+}$  and  $N_{\Pi_-}$ , we obtain an

analogous situation in which the sub-bundle  $L_-$  is integrable, i.e. it admits a foliation  $\mathcal{F}_-$  such that  $L_- = T\mathcal{F}_-$ , and in this case we call  $(M, K)$  an  $L_-$ -para-complex manifold.

Following the analogy with complex geometry, we introduce a compatible metric on almost para-complex manifolds, giving a counterpart of almost Hermitian manifolds.

**Definition 2.7.** An *almost para-Hermitian manifold*  $(M, K, \eta)$  is an almost para-complex manifold, i.e. a manifold  $M$  of even dimension  $2d$  endowed with a  $(1, 1)$ -tensor field  $K \in \text{End}(TM)$  such that  $K^2 = \mathbb{1}$ , together with a metric  $\eta$  of Lorentzian signature  $(d, d)$  which is compatible with the tensor  $K$  in the sense that

$$K^t \eta K = -\eta .$$

The compatibility condition can also be written as

$$\eta(K(X), K(Y)) = -\eta(X, Y) ,$$

or equivalently

$$\eta(K(X), Y) + \eta(X, K(Y)) = 0 , \quad (2.8)$$

for all  $X, Y \in \Gamma(TM)$ . The condition (2.8) implies that the distributions  $L_+$  and  $L_-$  are maximally isotropic with respect to  $\eta$ : For any  $X_+, Y_+ \in \Gamma(L_+)$ , we have  $K(X_+) = X_+$  and  $K(Y_+) = Y_+$ , and (2.8) gives  $\eta(X_+, Y_+) = 0$ , i.e.  $L_+$  is isotropic and, since  $\text{rank}(L_+) = d$ , it is a maximally isotropic sub-bundle of  $TM$ . The same argument applies to  $L_-$ .

From (2.8) we also deduce the existence of a non-degenerate 2-form field  $\omega$  on  $M$  given by

$$\omega(X, Y) = \eta(K(X), Y) ,$$

for all  $X, Y \in \Gamma(TM)$ , called the *fundamental 2-form*; it defines an *almost symplectic structure*, since it is generally not closed. Because of this definition, we have

$$\omega(X_+, Y_+) = 0 , \quad (2.9)$$

for all  $X_+, Y_+ \in \Gamma(L_+)$ , and

$$\omega(X_-, Y_-) = 0 , \quad (2.10)$$

for all  $X_-, Y_- \in \Gamma(L_-)$ . If the fundamental 2-form  $\omega$  is *symplectic*, i.e. it is moreover closed:  $d\omega = 0$ , then  $(M, K, \eta)$  is called an *almost para-Kähler manifold*. In this case, the conditions (2.9) and (2.10) imply that  $L_+$  and  $L_-$  are *Lagrangian* sub-bundles.

An *almost para-Hermitian structure*  $(K, \eta)$  on a manifold  $M$  can be regarded as a  $\mathbf{G}$ -structure on  $M$  given by a reduction of the structure group of  $TM$  from  $\text{GL}(2d, \mathbb{R})$  to the subgroup which preserves both  $\eta$  and  $\omega$ :

$$\mathbf{G} = \text{O}(d, d) \cap \text{Sp}(2d, \mathbb{R}) = \text{GL}(d, \mathbb{R}) .$$

**Interlude 2.11.** We denote by  $\text{O}(d, d)(M)$  the infinite-dimensional pseudo-orthogonal group of tangent bundle automorphisms  $\vartheta \in \text{End}(TM)$  which preserve the Lorentzian metric:  $\eta(\vartheta(X), \vartheta(Y)) = \eta(X, Y)$  for all  $X, Y \in \Gamma(TM)$ . This is the natural group of isometries of the almost para-Hermitian manifold  $(M, K, \eta)$  which is identified as its *continuous T-duality group*; any element  $\vartheta \in \text{O}(d, d)(M)$  can be regarded as a smooth map  $\vartheta : M \rightarrow \text{O}(d, d)$ . We denote by  $\text{SO}(d, d)(M)$  the Lie subgroup which also preserves the canonical orientation of  $M$  provided by its fundamental 2-form  $\omega$ ; its Lie

algebra  $\mathfrak{so}(d, d)(M)$  consists of endomorphisms  $\tau \in \text{End}(TM)$  such that  $\eta(\tau(X), Y) = -\eta(X, \tau(Y))$  for all  $X, Y \in \Gamma(TM)$ . Any element  $\tau \in \mathfrak{so}(d, d)(M)$  can be decomposed with respect to the splitting  $TM = L_+ \oplus L_-$  as

$$\tau = \begin{pmatrix} A & B_- \\ B_+ & -A^t \end{pmatrix},$$

where  $A \in \text{End}(L_+)$  with transpose  $A^t \in \text{End}(L_-)$  defined through  $\eta(A(X), Y) = \eta(X, A^t(Y))$ , while  $B_+ : \Gamma(L_+) \rightarrow \Gamma(L_-)$  and  $B_- : \Gamma(L_-) \rightarrow \Gamma(L_+)$  are skew morphisms in the sense that  $\eta(B_\pm(X), Y) = -\eta(X, B_\pm(Y))$ . By identifying  $L_-$  with  $L_+^*$  using the Lorentzian metric  $\eta$ , we can regard  $B_+$  as a 2-form  $B \in \bigwedge^2 \Gamma(L_+^*)$  called a *B-field* and  $B_-$  as a bivector  $\beta \in \bigwedge^2 \Gamma(L_+)$ , so that as a vector space

$$\mathfrak{so}(d, d)(M) = \text{End}(L_+) \oplus \bigwedge^2 \Gamma(L_+^*) \oplus \bigwedge^2 \Gamma(L_+).$$

Integrability of an almost para-Hermitian structure can be described as well. If the eigenbundles  $L_+$  and  $L_-$  of  $K$ , such that  $TM = L_+ \oplus L_-$ , are both integrable then the triple  $(M, K, \eta)$  is called a *para-Hermitian manifold*. If in addition the fundamental 2-form  $\omega = \eta K$  is closed, then  $(M, K, \eta)$  is said to be a *para-Kähler* (or *bi-Lagrangian*) *manifold*, in which case it has two transverse Lagrangian foliations with respect to the symplectic structure  $\omega$ .

In this framework, we can also describe partial integrability of the sub-bundles of  $TM$  and partial closure of the fundamental 2-form  $\omega$  (i.e.  $d\omega = 0$  on  $L_+$  or  $L_-$ ), giving rise to an assortment of new possible combinations. In particular, if  $(M, K, \eta)$  is an almost para-Hermitian manifold with an integrable sub-bundle  $L_+$ , then it is said to be an  *$L_+$ -para-Hermitian manifold*; there is an analogous definition replacing  $L_+$  with  $L_-$ . In general, it can be shown that

$$d\omega(\Pi_+(X), \Pi_+(Y), \Pi_+(Z)) = \sum_{(X, Y, Z)} \eta(N_{\Pi_+}(X, Y), Z),$$

where the sum runs over all cyclic permutations  $(X, Y, Z)$  of the three vector fields  $X, Y, Z \in \Gamma(TM)$ . This shows that non-integrability of the distribution  $L_+$  obstructs the closure of  $\omega$ ; in particular, integrability of  $L_+$  implies that the pullback of the 3-form  $d\omega$  to the foliation  $\mathcal{F}_+$  vanishes. In order to describe the geometry of such structures, a suitable connection is needed.

**Definition 2.12.** A *para-Hermitian connection*  $\nabla$  on an almost para-Hermitian manifold  $(M, K, \eta)$  is a connection on  $TM$  preserving  $\eta$  and  $\omega$ :

$$\nabla \eta = \nabla \omega = 0.$$

**Proposition 2.13.** Let  $(M, K, \eta)$  be an almost para-Hermitian manifold with fundamental 2-form  $\omega$ , and let  $\nabla^{\text{LC}}$  be the Levi-Civita connection of  $\eta$ . Then the covariant Levi-Civita derivative of  $\omega$  satisfies

$$\nabla_X^{\text{LC}} \omega(\Pi_+(Y), \Pi_-(Z)) = 0, \tag{2.14}$$

$$\nabla_X^{\text{LC}} \omega(Y, Z) = \eta(\nabla_X^{\text{LC}} K(Y), Z), \tag{2.15}$$

$$d\omega(X, Y, Z) = \sum_{(X, Y, Z)} \nabla_X^{\text{LC}} \omega(Y, Z), \tag{2.16}$$

for all  $X, Y, Z \in \Gamma(TM)$ .

The proof can be found in [16]. This proposition stresses the role of the Levi-Civita connection  $\nabla^{\text{LC}}$  in para-Hermitian geometry. In particular, it implies that  $\nabla^{\text{LC}}$  is a para-Hermitian connection if and only if  $(M, K, \eta)$  is an almost para-Kähler manifold.<sup>3</sup>

On any almost para-Hermitian manifold  $(M, K, \eta)$ , the *canonical para-Hermitian connection* is obtained from the Levi-Civita connection as

$$\nabla^{\text{can}} = \Pi_+ \nabla^{\text{LC}} \Pi_+ + \Pi_- \nabla^{\text{LC}} \Pi_- .$$

Equivalently, the canonical connection  $\nabla^{\text{can}}$  is defined as the connection satisfying

$$\eta(\nabla_X^{\text{can}} Y, Z) = \eta(\nabla_X^{\text{LC}} Y, Z) - \frac{1}{2} \nabla_X^{\text{LC}} \omega(Y, K(Z)) ,$$

for all vector fields  $X, Y, Z$ . This connection is the key ingredient for the study of the D-bracket, which we introduce below since it gives a new interpretation of the generalized fluxes in double field theory [19].

**Interlude 2.17.** The splitting of the tangent bundle  $TM$  gives rise to a decomposition of tensors analogous to the type decomposition in complex geometry. In particular, there is such a decomposition for differential forms. We denote  $\bigwedge^{(+k, -0)} \Gamma(T^*M) = \bigwedge^k \Gamma(L_+^*)$  and  $\bigwedge^{(+0, -k)} \Gamma(T^*M) = \bigwedge^k \Gamma(L_-^*)$ , so that any  $k$ -form on  $M$  is decomposed according to the splitting

$$\bigwedge^k \Gamma(T^*M) = \bigoplus_{m+n=k} \bigwedge^{(+m, -n)} \Gamma(T^*M) .$$

The fundamental 2-form  $\omega$  of an almost para-Hermitian manifold is a  $(+1, -1)$ -form with respect to the almost para-Hermitian structure  $(K, \eta)$ , since both  $L_+$  and  $L_-$  are Lagrangian with respect to  $\omega$ .

## 2.2 Brackets and Algebroids

As discussed in [16, 19], the D-bracket is needed to formulate a precise relation between  $L_{\pm}$ -para-Hermitian manifolds and (standard) Courant algebroids. The definition of a particular D-bracket also gives a global formulation of the D-bracket of double field theory.

Let us first describe how a bracket on vector fields can be associated to any connection on an almost para-Hermitian manifold.

**Definition 2.18.** Let  $(M, K, \eta)$  be an almost para-Hermitian manifold and  $\nabla$  a connection on  $TM$ . The *bracket*  $\llbracket \cdot, \cdot \rrbracket^{\nabla}$  associated to  $\nabla$  is defined by

$$\eta(\llbracket X, Y \rrbracket^{\nabla}, Z) = \eta(\nabla_X Y - \nabla_Y X, Z) + \eta(\nabla_Z X, Y) , \quad (2.19)$$

and the  $\Pi_{\pm}$ -projected brackets  $\llbracket \cdot, \cdot \rrbracket_{\pm}^{\nabla}$  associated to  $\nabla$  by

$$\eta(\llbracket X, Y \rrbracket_{\pm}^{\nabla}, Z) = \eta(\nabla_{\Pi_{\pm}(X)} Y - \nabla_{\Pi_{\pm}(Y)} X, Z) + \eta(\nabla_{\Pi_{\pm}(Z)} X, Y) , \quad (2.20)$$

for all  $X, Y, Z \in \Gamma(TM)$ .

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<sup>3</sup>In this sense, para-Kähler manifolds provide the closest versions of the conventional flat space formulations of double field theory in the absence of fluxes.

Since  $\Pi_+ + \Pi_- = \mathbb{1}$ , it follows that

$$[[X, Y]]^\nabla = [[X, Y]]_+^\nabla + [[X, Y]]_-^\nabla$$

for all  $X, Y \in \Gamma(TM)$ . The second term on the right-hand side in both (2.19) and (2.20) is usually called a *twist*. In particular, for a torsion-free connection  $\nabla$ , the associated bracket is given by the Lie bracket  $[X, Y]$  of vector fields plus a twist term determined by the connection itself. The covariant derivative  $\nabla_{\Pi_\pm(X)}$  is also called a *projected derivative* and it gives a *projected differential*  $D : C^\infty(M) \rightarrow \Gamma(TM)$  defined by  $\eta(X, Df) = \Pi_\pm(X)(f)$  for all  $X \in \Gamma(TM)$  and  $f \in C^\infty(M)$ . The projected derivative is a derivation, i.e. it satisfies the Leibniz rule

$$\nabla_{\Pi_\pm(X)}(fY) = \Pi_\pm(X)(f)Y + f\nabla_{\Pi_\pm(X)}Y .$$

A full Riemannian characterization of the projected geometry is given in [16], including the projected Riemann tensor and projected torsion.

In order to extend what we did so far, we introduce a new class of brackets by weakening the compatibility condition following [14, 15].

**Definition 2.21.** Let  $(M, K, \eta)$  be an almost para-Hermitian manifold. A *metric-compatible bracket* is a bilinear operation  $[[\cdot, \cdot]] : \Gamma(TM) \times \Gamma(TM) \rightarrow \Gamma(TM)$  satisfying

$$\begin{aligned} X(\eta(Y, Z)) &= \eta([X, Y], Z) + \eta(Y, [X, Z]) , \\ [[X, fY]] &= f[[X, Y]] + X(f)Y , \\ \eta(Y, [[X, X]]) &= \eta([Y, X], X) , \end{aligned}$$

for all  $X, Y, Z \in \Gamma(TM)$  and  $f \in C^\infty(M)$ .

The triple  $(TM, \eta, [[\cdot, \cdot]])$  defines a *metric algebroid* with anchor given by the identity map. Note that the anchor is not required to satisfy any compatibility condition between the metric-compatible bracket and the Lie bracket; metric algebroids are related to the pre-DFT algebroids of [18] that naturally arise in the extensions of the Courant algebroids of generalized geometry to double field theory.

Following [17], we introduce a generalized notion of integrability for a metric algebroid.

**Definition 2.22.** Let  $(TM, \eta, [[\cdot, \cdot]])$  be a metric algebroid and  $K \in \text{End}(TM)$  a tensor field such that

$$K^2 = \pm \mathbb{1} \quad \text{and} \quad \eta(K(X), Y) = -\eta(X, K(Y)) ,$$

for all  $X, Y \in \Gamma(TM)$ . The *generalized Nijenhuis tensor*  $\mathcal{N}_K : \Gamma(TM) \times \Gamma(TM) \rightarrow \Gamma(TM)$  associated with  $K$  is given by

$$\mathcal{N}_K(X, Y) = K^2([X, Y]) + [[K(X), K(Y)]] - K([K(X), Y] + [X, K(Y)]) .$$

This allows one to define a D-bracket in the context of para-Hermitian geometry.

**Definition 2.23.** A *D-bracket* on an almost para-Hermitian manifold  $(M, K, \eta)$  is a metric-compatible bracket  $[[\cdot, \cdot]]$  for which the generalized Nijenhuis tensor associated with  $K$  vanishes:  $\mathcal{N}_K(X, Y) =$

0 for all  $X, Y \in \Gamma(TM)$ . The triple  $(K, \eta, \llbracket \cdot, \cdot \rrbracket)$  is a *D-structure* on  $M$ . A D-structure is *canonical* if it satisfies

$$\llbracket \Pi_+(X), \Pi_+(Y) \rrbracket = \Pi_+([\Pi_+(X), \Pi_+(Y)]) \quad \text{and} \quad \llbracket \Pi_-(X), \Pi_-(Y) \rrbracket = \Pi_-([\Pi_-(X), \Pi_-(Y)]) .$$

It is shown in [17] that there exists a unique canonical D-bracket on  $(M, K, \eta)$  which is given by the bracket defined in (2.19) associated with the canonical para-Hermitian connection  $\nabla^{\text{can}}$ . We will refer to this bracket as the *D-bracket associated with  $K$* , and denote it by  $\llbracket \cdot, \cdot \rrbracket^{\text{D}}$ .

For a flat manifold the D-bracket has the local expression of the D-bracket of double field theory, as shown in [14]. The *C-bracket* is obtained as the skew-symmetrization of the D-bracket:

$$\llbracket X, Y \rrbracket^{\text{C}} = \frac{1}{2} (\llbracket X, Y \rrbracket^{\text{D}} - \llbracket Y, X \rrbracket^{\text{D}}) . \quad (2.24)$$

It is shown in [19] that the relation between the D-bracket and the bracket associated to the Levi-Civita connection, the  $\nabla^{\text{LC}}$ -bracket, involves the exterior derivative of the fundamental 2-form  $\omega$ , such that the D-bracket of an almost para-Hermitian manifold is given by

$$\begin{aligned} \eta(\llbracket X, Y \rrbracket^{\text{D}}, Z) &= \eta(\llbracket X, Y \rrbracket^{\nabla^{\text{LC}}}, Z) \\ &\quad - \frac{1}{2} (\text{d}\omega^{(+3, -0)} + \text{d}\omega^{(+2, -1)} - \text{d}\omega^{(+0, -3)} - \text{d}\omega^{(+1, -2)})(X, Y, Z) . \end{aligned} \quad (2.25)$$

The difference between a D-bracket and a  $\nabla^{\text{LC}}$ -bracket is also called *generalized torsion* of the  $\nabla^{\text{LC}}$ -bracket. For an almost para-Kähler manifold,  $\nabla^{\text{can}} = \nabla^{\text{LC}}$  and hence  $\llbracket X, Y \rrbracket^{\text{D}} = \llbracket X, Y \rrbracket^{\nabla^{\text{LC}}}$ .

From this discussion there also emerges a new interpretation of the section condition of double field theory, proven in [19].

**Proposition 2.26.** Let  $(M, K, \eta)$  be a flat para-Hermitian manifold and let  $X_+, Y_+ \in \Gamma(L_+)$  be vector fields which are parallel along  $\mathcal{F}_-$ . Then

$$\llbracket X_+, Y_+ \rrbracket^{\text{D}} = \llbracket X_+, Y_+ \rrbracket_+^{\text{D}} ,$$

or equivalently

$$\llbracket X_+, Y_+ \rrbracket_-^{\text{D}} = 0 .$$

Here we see that the section condition  $\llbracket X_+, Y_+ \rrbracket_-^{\text{D}} = 0$  restricts the vector fields to be sections over the foliation  $\mathcal{F}_+$ .

## 2.3 Flux Deformations of Para-Hermitian Structures

We shall now define special isometries relating two different almost para-Hermitian structures on the same manifold  $M$  and describe how the D-bracket transforms under their action. In this description we will see strong similarities with the transformations proposed in generalized geometry [1, 2]. We will find that some geometric and non-geometric fluxes appear in this discussion as obstructions to a weaker notion of integrability.

We first need the notion of *B-transformation* for an almost para-Hermitian manifold.

**Definition 2.27.** Let  $(M, K, \eta)$  be an almost para-Hermitian manifold. A  *$B_+$ -transformation* is

an isometry of  $TM$  given by

$$e^{B_+} = \begin{pmatrix} \mathbb{1} & 0 \\ B_+ & \mathbb{1} \end{pmatrix} \in \text{O}(d, d)(M) ,$$

where we have chosen the splitting  $TM = L_+ \oplus L_-$  and  $B_+ : \Gamma(L_+) \rightarrow \Gamma(L_-)$  is a skew map in the sense that it satisfies  $\eta(B_+(X), Y) = -\eta(X, B_+(Y))$ .

A  $B_+$ -transformation of the almost para-complex structure  $K$  is then given by

$$K \longmapsto K_{B_+} = e^{B_+} K e^{-B_+} .$$

In the splitting  $TM = L_+ \oplus L_-$ , the tensor  $K$  is given by

$$K = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix} ,$$

and hence the transformed almost para-complex structure takes the form

$$K_{B_+} = \begin{pmatrix} \mathbb{1} & 0 \\ 2B_+ & -\mathbb{1} \end{pmatrix} . \quad (2.28)$$

One easily has  $K_{B_+}^2 = \mathbb{1}$ , while the skew property of  $B_+$  is required for the compatibility condition  $\eta(K_{B_+}(X), K_{B_+}(Y)) = -\eta(X, Y)$  to be satisfied.

The endomorphism  $B_+$  is given either by a 2-form  $b_+$  or by a bivector  $\beta_-$  defined by

$$\eta(B_+(X), Y) = b_+(X, Y) = \beta_-(\eta(X), \eta(Y)) .$$

The 2-form  $b_+$  is of type  $(+2, -0)$ , while the bivector  $\beta_-$  is of type  $(+0, -2)$  with respect to  $K$ . This is relevant to understanding how the fundamental 2-form  $\omega$  changes under a  $B_+$ -transformation:

$$\omega \longmapsto \omega_{B_+} = \eta K_{B_+} \omega = \omega + 2b_+ ,$$

so that such transformations may not preserve the closure of the fundamental 2-form. A completely analogous discussion can be carried out for a  $B_-$ -transformation, defined by a skew map  $B_- : \Gamma(L_-) \rightarrow \Gamma(L_+)$ .

The main effect of a  $B_+$ -transformation is that the splitting  $TM = L_+ \oplus L_-$  changes, i.e.  $e^{B_+} : L_+ \oplus L_- \rightarrow L_+^{B_+} \oplus L_-^{B_+}$ , which implies that the potential Frobenius integrability of the original splitting may not be preserved in its image under  $e^{B_+}$ . The transformed projections are given by

$$\Pi_+^{B_+} = \frac{1}{2} (\mathbb{1} + K_{B_+}) = \begin{pmatrix} \mathbb{1} & 0 \\ B_+ & 0 \end{pmatrix} \quad \text{and} \quad \Pi_-^{B_+} = \frac{1}{2} (\mathbb{1} - K_{B_+}) = \begin{pmatrix} 0 & 0 \\ -B_+ & \mathbb{1} \end{pmatrix} .$$

Hence, decomposing any vector field as

$$X = \begin{pmatrix} X_+ \\ X_- \end{pmatrix} \in \Gamma(TM) ,$$

where  $X_+ \in \Gamma(L_+)$  and  $X_- \in \Gamma(L_-)$ , the new distributions are obtained by using the transformed



projections to get

$$\Pi_+^{B+}(X) = X_+ + B_+(X_+) \quad \text{and} \quad \Pi_-^{B+}(X) = X_- - B_+(X_+) ,$$

where  $\Pi_-^{B+}(X) \in \Gamma(L_-)$  since  $B_+$  maps  $\Gamma(L_+)$  to  $\Gamma(L_-)$ . Thus  $L_-^{B+} = L_-$ . On the other hand, the same reasoning applied to  $\Pi_+^{B+}(X)$  shows that it is not an element of  $\Gamma(L_+)$ , i.e.  $L_+^{B+} \neq L_+$ . Therefore only the  $-1$ -eigenbundle is preserved by a  $B_+$ -transformation, while the  $+1$ -eigenbundle changes; in particular, if  $L_+$  is integrable, then integrability of  $L_+^{B+}$  is generally violated.

In order to compare two different almost para-Hermitian structures on the same manifold, a weaker notion of integrability is introduced. The main difference from the usual notion of Frobenius integrability is the replacement of the Lie bracket of vector fields with the D-bracket.

**Definition 2.29.** Let  $(M, K, \eta)$  be an almost para-Hermitian manifold with associated D-bracket  $\llbracket \cdot, \cdot \rrbracket^{\mathbb{D}}$ . An isotropic (with respect to  $\eta$ ) distribution  $\mathcal{D}$  is *weakly integrable* if it is involutive under the D-bracket:

$$\llbracket \Gamma(\mathcal{D}), \Gamma(\mathcal{D}) \rrbracket^{\mathbb{D}} \subseteq \Gamma(\mathcal{D}) .$$

For example, the eigenbundles  $L_+$  and  $L_-$  of  $K$  are always weakly integrable. It is clear from this definition that the notion of weak integrability depends on the choice of the almost para-Hermitian structure, as this choice represents the reference almost para-Hermitian structure which defines the D-bracket. Hence we can formulate a notion of compatibility based on this relative integrability.

**Definition 2.30.** Let  $(K, \eta)$  and  $(K', \eta)$  be two almost para-Hermitian structures on a manifold  $M$ . Then  $K'$  is *compatible with  $K$*  if the eigenbundles of  $K'$  are weakly integrable with respect to  $K$ .

Any almost para-complex structure  $K$  is always compatible with itself. We can thus analyze the weak integrability of a  $B_+$ -transformed almost para-complex structure  $K_{B_+}$  with respect to the original structure  $K$ . For this, we note that the D-bracket of sections of the  $+1$ -eigenbundle of  $K_{B_+}$  is given by

$$\eta(\llbracket \Pi_+^{B+}(X), \Pi_+^{B+}(Y) \rrbracket^{\mathbb{D}}, \Pi_+^{B+}(Z)) = \left( d_+ b_+ + (\wedge^3 \eta) [\beta_-, \beta_-]_-^{\mathbb{S}} \right) (\Pi_+^{B+}(X), \Pi_+^{B+}(Y), \Pi_+^{B+}(Z)) ,$$

where  $d_+$  is the Lie algebroid differential of  $L_+$  and  $[\cdot, \cdot]_-^{\mathbb{S}}$  is the Schouten-Nijenhuis bracket of  $L_-$ . The  $+1$ -eigenbundle of  $K_{B_+}$  is weakly integrable if and only if

$$\eta(\llbracket \Pi_+^{B+}(X), \Pi_+^{B+}(Y) \rrbracket^{\mathbb{D}}, \Pi_+^{B+}(Z)) = 0 .$$

This implies that the Maurer-Cartan equation

$$d_+ b_+ + (\wedge^3 \eta) [\beta_-, \beta_-]_-^{\mathbb{S}} = 0 \tag{2.31}$$

has to be satisfied in order for  $K_{B_+}$  to be compatible with  $K$ . If this equation is not satisfied, then the components of  $d_+ b_+$  can be interpreted as fluxes, as shown in [19], with  $d_+ b_+ + (\wedge^3 \eta) [\beta_-, \beta_-]_-^{\mathbb{S}}$  giving the covariant NS-NS  $H$ -flux. A  $B_-$ -transformation gives the corresponding dual non-geometric  $R$ -flux.

It is shown in [19] that the D-bracket  $\llbracket \cdot, \cdot \rrbracket_{B_+}^{\mathbb{D}}$  associated to  $K_{B_+}$  is related to the original D-bracket associated to an almost para-Kähler structure  $K$  by

$$\eta(\llbracket X, Y \rrbracket_{B_+}^{\mathbb{D}}, Z) = \eta(\llbracket X, Y \rrbracket^{\mathbb{D}}, Z) - db_+(X, Y, Z) , \tag{2.32}$$

where the  $(+3, -0)$ -component of  $db_+$  (with respect to the splitting defined by  $K_{B_+}$ ) coincides with the left-hand side of (2.31), while the  $(+2, -1)$ -component gives the dual non-geometric  $Q$ -flux. In this framework, the geometric  $f$ -flux also arises generally via the D-bracket  $\llbracket \cdot, \cdot \rrbracket^D$  in the usual way through diagonal isometries of the tangent bundle  $TM = L_+ \oplus L_-$ ,

$$\begin{pmatrix} A & 0 \\ 0 & (A^{-1})^t \end{pmatrix} \in \mathcal{O}(d, d)(M), \quad (2.33)$$

with  $A \in \text{End}(L_+)$ , which preserve  $K$  and rotate frames on the sub-bundles  $L_+$  and  $L_-$ ; here  $L_-$  is identified with  $L_+^*$  using the Lorentzian metric  $\eta$ .

## 2.4 Recovering the Physical Spacetime Background

Let us now briefly describe the physical interpretation of the formalism thus far, in particular how para-Hermitian geometry recovers the usual expectations of the doubled geometry of double field theory and reconciles them with generalized geometry. Building on the local description in [4], an almost para-Hermitian structure  $(K, \eta)$  on a  $2d$ -dimensional manifold  $M$ , i.e. a splitting of the tangent bundle  $TM = L_+ \oplus L_-$  into maximally isotropic sub-bundles, is also called a *polarization*. To make contact with the generalized geometry of the standard Courant algebroid, only one of the two distributions of  $TM$  is required to be integrable, in which case  $(M, K, \eta)$  is not almost para-Kähler (equivalently  $\nabla^{\text{LC}}\omega \neq 0$ ). If  $(M, K, \eta)$  is an  $L_+$ -para-Hermitian manifold, then  $L_+ = T\mathcal{F}_+$  is the tangent bundle of a foliation  $\mathcal{F}_+$  of  $M$  of dimension  $d$ ; the Lagrangian submanifold  $\mathcal{F}_+$  is then the physical spacetime.<sup>4</sup> The tangent bundle  $TM$  can be regarded as a metric algebroid over  $\mathcal{F}_+$  with anchor map given by the  $L_+$ -para-complex projection  $\Pi_+ : \Gamma(TM) \rightarrow \Gamma(T\mathcal{F}_+)$ . The  $\mathcal{O}(d, d)(M)$ -invariant metric  $\eta$  identifies  $L_-$  with  $L_+^* = T^*\mathcal{F}_+$ , and  $TM$  is identified with the generalized tangent bundle<sup>5</sup>  $\mathbb{T}\mathcal{F}_+ := T\mathcal{F}_+ \oplus T^*\mathcal{F}_+$  of  $\mathcal{F}_+$  via the projection isomorphism

$$\mathbf{p}_+ : \Gamma(TM) \longrightarrow \Gamma(\mathbb{T}\mathcal{F}_+) , \quad X \longmapsto \mathbf{p}_+(X) = \Pi_+(X) + \eta(\Pi_-(X)) ,$$

with inverse  $\mathbf{p}_+^{-1} : X_+ + \alpha_+ \mapsto X_+ + \eta^{-1}(\alpha_+)$  for  $X_+ \in \Gamma(T\mathcal{F}_+)$  and  $\alpha_+ \in \Gamma(T^*\mathcal{F}_+)$ . This isomorphism sends the metric  $\eta$  and the fundamental 2-form  $\omega$  to the duality pairings

$$\begin{aligned} \eta(\mathbf{p}_+^{-1}(X_+ + \alpha_+), \mathbf{p}_+^{-1}(Y_+ + \beta_+)) &= \beta_+(X_+) + \alpha_+(Y_+) , \\ \omega(\mathbf{p}_+^{-1}(X_+ + \alpha_+), \mathbf{p}_+^{-1}(Y_+ + \beta_+)) &= \beta_+(X_+) - \alpha_+(Y_+) . \end{aligned}$$

It also sends the canonical D-structure on  $M$  to the standard Courant algebroid over  $\mathcal{F}_+$ , with the  $\Pi_+$ -projected D-bracket  $\llbracket \cdot, \cdot \rrbracket_+^D$  on sections of  $TM$  mapping to the Dorfman bracket  $[\cdot, \cdot]_{\mathcal{F}_+}^D$  on sections of  $\mathbb{T}\mathcal{F}_+$ :

$$\mathbf{p}_+(\llbracket X, Y \rrbracket_+^D) = [\mathbf{p}_+(X), \mathbf{p}_+(Y)]_{\mathcal{F}_+}^D .$$

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<sup>4</sup>If  $(M, K, \eta)$  is a para-Hermitian manifold, so that  $L_- = T\mathcal{F}_-$  is also integrable, then the foliation  $\mathcal{F}_-$  may be interpreted as the auxiliary “dual” manifold of the physical spacetime  $\mathcal{F}_+$ . However, this situation will only arise in some very special instances in this paper and is not a general feature of a global doubled geometry.

<sup>5</sup>Globally, the generalized tangent bundle on  $\mathcal{F}_+$  is the vector bundle  $E \rightarrow \mathcal{F}_+$  defined by the exact sequence

$$0 \longrightarrow T^*\mathcal{F}_+ \longrightarrow E \longrightarrow T\mathcal{F}_+ \longrightarrow 0$$

of bundles on  $\mathcal{F}_+$ , where the map  $\rho : E \rightarrow T\mathcal{F}_+$  is an anchor. This global description must be used whenever  $\mathcal{F}_+$  is endowed with a non-trivial NS-NS  $H$ -flux.

This result is established in detail in [16, 17, 19]; an analogous result is given by [18] in terms of the C-bracket (2.24) on  $TM$  and the Courant bracket on  $\mathbb{T}\mathcal{F}_+$  for the particular case where  $M = T^*\mathcal{Q}$  is the total space of the cotangent bundle of a  $d$ -dimensional manifold  $\mathcal{Q}$ , which in subsequent sections we will describe as a key example of an almost para-Hermitian manifold.

In order to lead to a string background, we also need to specify how to recover the physical background fields of supergravity on  $\mathcal{F}_+$ , such as the spacetime metric  $g$  and the Kalb-Ramond field  $B$ . This requires a dynamical augmentation of the kinematical data of an almost para-Hermitian structure, for which we follow [17, 25].

**Definition 2.34.** A *generalized metric* on an almost para-Hermitian manifold  $(M, K, \eta)$  is a Riemannian metric  $\mathcal{H}$  on  $M$  which is compatible with the  $\mathrm{O}(d, d)(M)$ -invariant metric  $\eta$  and the fundamental 2-form  $\omega$  in the sense that

$$\eta^{-1} \mathcal{H} = \mathcal{H}^{-1} \eta \quad \text{and} \quad \omega^{-1} \mathcal{H} = -\mathcal{H}^{-1} \omega .$$

The triple  $(\eta, \omega, \mathcal{H})$  is a *Born geometry* on  $M$  and  $(M, \eta, \omega, \mathcal{H})$  is a *Born manifold*.

A generalized metric can also be regarded as an almost Hermitian metric relative to  $\omega$ , while a Born geometry can be regarded as a  $\mathbf{G}$ -structure on  $M$  with

$$\mathbf{G} = \mathrm{O}(d, d) \cap \mathrm{Sp}(2d, \mathbb{R}) \cap \mathrm{O}(2d) = \mathrm{O}(d) .$$

It is shown in [17] that there always exists a choice of frame on  $TM = L_+ \oplus L_-$  in which the generalized metric can be brought into the diagonal form

$$\mathcal{H}_0 = \begin{pmatrix} g_+ & 0 \\ 0 & g_+^{-1} \end{pmatrix} , \tag{2.35}$$

where  $g_+$  is a metric on the sub-bundle  $L_+$  and we have identified  $L_- = L_+^*$  using the Lorentzian metric  $\eta$ . In the case that  $(M, K, \eta)$  is  $L_+$ -para-Hermitian, this shows that the generalized metric encodes a choice of Riemannian metric on the physical spacetime submanifold  $\mathcal{F}_+$  and provides an  $\mathrm{O}(d) \times \mathrm{O}(d)$ -structure on the generalized tangent bundle  $\mathbb{T}\mathcal{F}_+$ .

Having discussed how to obtain the conventional spacetime description from a choice of polarization, it is then natural to understand the meaning of changing polarization in the framework of para-Hermitian geometry, extending the notions introduced in [4].

**Definition 2.36.** A *change of polarization* on an almost para-Hermitian manifold  $(M, K, \eta)$  is an isometry  $\vartheta \in \mathrm{O}(d, d)(M)$  mapping the almost para-Hermitian structure  $(K, \eta)$  into  $(K_\vartheta, \eta)$  with  $K_\vartheta = \vartheta^{-1} K \vartheta$ .

From this definition it is easy to check that  $(K_\vartheta, \eta)$  is also an almost para-Hermitian structure on  $M$ , i.e.  $K_\vartheta^2 = \mathbb{1}$  and  $K_\vartheta^\mathrm{t} \eta K_\vartheta = -\eta$ , and that the fundamental 2-form transforms into

$$\omega \longmapsto \omega_\vartheta = \eta K_\vartheta = \vartheta^\mathrm{t} \omega \vartheta .$$

Such transformations do not generally preserve the (Frobenius or weak) integrability of the eigendistributions, or the closure of the fundamental 2-form. In this sense, the choice of polarization contains all information about fluxes and the spacetime background. We will show explicitly later on,

through some prototypical examples, that the fluxes appear as obstructions to weak integrability with respect to a reference para-Kähler structure. On the other hand, the background geometry arises from a choice of generalized metric  $\mathcal{H}$ , i.e. a Born geometry  $(\eta, \omega, \mathcal{H})$  on  $M$ , which transforms under a change of polarization into

$$\mathcal{H} \longmapsto \mathcal{H}_\vartheta = \vartheta^t \mathcal{H} \vartheta .$$

Importantly, these transformations describe T-dualities (and other symmetries) on an almost para-Hermitian manifold, interpreted as a doubled spacetime: The smooth map  $\vartheta : M \rightarrow \mathcal{O}(d, d)$  acts by an element of the continuous T-duality group  $\mathcal{O}(d, d)$ . We have already encountered a special class of changes of polarization, namely the  $B$ -transformations of Section 2.3. In this case, a  $B_+$ -transformation changes the polarization by  $\vartheta = e^{-B_+}$ ; in particular, the diagonal generalized metric (2.35) is mapped to

$$\mathcal{H}_{B_+} = (e^{-B_+})^t \mathcal{H}_0 e^{-B_+} = \begin{pmatrix} g_+ - b_+ g_+^{-1} b_+ & b_+ g_+^{-1} \\ -g_+^{-1} b_+ & g_+^{-1} \end{pmatrix} . \quad (2.37)$$

When  $(M, K, \eta)$  is an  $L_+$ -para-Hermitian manifold, this is the familiar form from generalized geometry of the generalized metric on the physical spacetime  $\mathcal{F}_+$  which unifies the target space metric  $g_+$  and the Kalb-Ramond 2-form field  $b_+ = \eta B_+$ .

For later use, let us spell out the form of such transformations in local coordinates, and in particular show that any two almost para-Hermitian structures on the same manifold  $M$  with the same compatible metric  $\eta$  are related by a change of polarization in the sense of Definition 2.36. Let  $(M, K, \eta)$  be an almost para-Hermitian manifold whose eigendistributions  $L_+$  and  $L_-$  are locally spanned, in a given open contractible chart on  $M$ , by vector fields  $Z_i$  and  $\tilde{Z}^i$ :  $\Gamma(L_+) = \text{Span}_{C^\infty(M)}\{Z_i\}$  and  $\Gamma(L_-) = \text{Span}_{C^\infty(M)}\{\tilde{Z}^i\}$ . Let  $\Theta^i$  and  $\tilde{\Theta}_i$  be the respective dual 1-forms, so that we can write the local expression of the almost para-Hermitian structure as<sup>6</sup>

$$K = Z_i \otimes \Theta^i - \tilde{Z}^i \otimes \tilde{\Theta}_i \quad \text{and} \quad \eta = \eta_i^j (\Theta^i \otimes \tilde{\Theta}_j + \tilde{\Theta}_j \otimes \Theta^i)$$

with  $\omega = \eta_j^i \tilde{\Theta}_i \wedge \Theta^j$ . Given another almost para-Hermitian structure  $(K', \eta)$  on  $M$ , we write the corresponding eigendistributions locally in the same chart as  $\Gamma(L'_+) = \text{Span}_{C^\infty(M)}\{Z'_i\}$  and  $\Gamma(L'_-) = \text{Span}_{C^\infty(M)}\{\tilde{Z}'^i\}$ , with dual 1-forms  $\Theta'^i$  and  $\tilde{\Theta}'_i$ . We can then write the  $\mathcal{O}(d, d)(M)$ -transformation from  $K$  to  $K'$  as

$$\vartheta = Z_i \otimes \Theta'^i + \tilde{Z}^i \otimes \tilde{\Theta}'_i ,$$

whose inverse is thus given by

$$\vartheta^{-1} = Z'_i \otimes \Theta^i + \tilde{Z}'^i \otimes \tilde{\Theta}_i ,$$

and whose transpose is

$$\vartheta^t = \tilde{\Theta}'_i \otimes \tilde{Z}^i + \Theta'^i \otimes Z_i .$$

It is easily checked that

$$\eta' = \vartheta^t \eta \vartheta = \eta_i^j (\Theta'^i \otimes \tilde{\Theta}'_j + \tilde{\Theta}'_j \otimes \Theta'^i) = \eta ,$$

which is indeed compatible with  $K'$ .

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<sup>6</sup>Throughout implicit summation over repeated upper and lower indices is understood.

The fact that changes of polarization generally induce flux deformations of the almost para-Hermitian structure, and hence may spoil (Frobenius or weak) integrability of the eigendistributions, means that a conventional spacetime description is not always possible [35]. While geometric fluxes give twisted distributions which are globally (weakly) integrable, some flux deformations preserve weak integrability only locally and the foliations are not globally defined; such fluxes are said to be *globally non-geometric*. Others spoil integrability altogether, so that not even a local geometric spacetime picture can emerge; such fluxes are called *locally non-geometric*. In the following we will spell this picture out explicitly in several concrete classes of backgrounds, and in particular obtain a new geometric impetus on the point of view that non-geometric backgrounds are noncommutative and nonassociative spacetimes [31, 32, 37–44]. In the case where the polarizations are related by T-dualities or other symmetries of string theory, they give physically equivalent string backgrounds. In this paper we do not address the general problem of which changes of polarization  $\vartheta \in \mathcal{O}(d, d)(M)$  yield proper string symmetries.

### 3 Dynamical Para-Kähler Structures

In the following we will describe some dynamical systems in which para-Hermitian structures naturally arise, giving a more elementary appearance of para-Hermitian geometry than in the construction of a globally well-defined setting for the kinematics of double field theory. A clarifying class of examples of para-Hermitian geometry comes from Lagrangian dynamics, i.e. from the tangent lift of the dynamics to the tangent bundle of a configuration space with a sufficiently regular function defined on this bundle which encodes the equations of motion. In this section we give a new interpretation to a widely discussed subject, commonly known as Finsler geometry which describes the geometry arising from regular functions on a manifold (such as Lagrangians and Hamiltonians), as an instance of para-Kähler geometry; a discussion of Lagrangian and Hamiltonian geometry in terms of Finsler geometry can be found in [45].

#### 3.1 Newtonian Dynamical Systems and Their Lifts

In order to understand the lifting procedure, we need a precise definition of a dynamical system. In this paper we will focus on Newtonian dynamical systems [46].

**Definition 3.1.** A *Newtonian dynamical system* is given by a  $d$ -dimensional manifold  $\mathcal{Q}$ , called *configuration space*, and a second order differential equation given, in a local chart of  $\mathcal{Q}$  with coordinates  $q = (q^i)$ , by

$$\frac{d^2 q^i}{dt^2} = \Phi^i(q, \dot{q}) , \quad (3.2)$$

with  $t$  a real parameter,  $\dot{q}^i = \frac{dq^i}{dt}$  and  $\Phi^i(q, \dot{q})$  a function of  $(q^i, \dot{q}^i)$  assigning a time evolution law. A *trajectory* of the dynamical system is a curve  $g : \mathbb{R} \rightarrow \mathcal{Q}$  given, in a local chart  $(U, \phi) = (U, q^i)$  on  $\mathcal{Q}$ , by  $\phi \circ g : \mathbb{R} \ni t \mapsto (q^i(t)) \in \mathbb{R}^d$  such that  $q^i(t)$  are solutions of the differential equation (3.2).

The differential equation (3.2) does not separate the trajectories on  $\mathcal{Q}$ , i.e. there are an infinite number of trajectories passing through each point in  $\mathcal{Q}$ , and hence a different description of the dynamical system is needed in order to find a unique solution to (3.2) for any set of initial conditions. Roughly speaking, we need to find an equivalent system of *first order* differential equations by

enlarging the space on which they are defined so that there are enough initial conditions to formulate a well-posed Cauchy problem, and hence to obtain a unique solution. From a geometric point of view, this means that we have to find, on this enlarged manifold  $M$ , a vector field  $\Sigma \in \Gamma(TM)$  with components locally defined by first order differential equations, whose integral curves can be *projected* onto the trajectories of the dynamical system on  $\mathcal{Q}$ . This leads to the definition of a lift of the dynamics.

**Definition 3.3.** A *lift of the dynamics* is the association of an equivalent first order dynamical system field  $\Sigma \in \Gamma(TM)$  on a *carrier manifold*  $M$  to the Newtonian dynamical system on  $\mathcal{Q}$ . The inverse procedure of mapping integral curves of the first order dynamical system field  $\Sigma$  to trajectories of the original system is *projection*.

This definition shows that fiber bundles with base space the configuration space  $\mathcal{Q}$  are natural choices for lifting the dynamics. In particular, the bundle projection plays a crucial role in the description of the geometry of such lifts: If we consider as carrier manifold  $M$  the total space of a fiber bundle  $E$  with smooth structure induced by that of the base manifold  $\mathcal{Q}$ , the projection is naturally defined by the surjective map  $\pi : E \rightarrow \mathcal{Q}$ . The tangent map  $T\pi : TE \rightarrow T\mathcal{Q}$  induced by the projection defines a splitting  $TE = L_v(E) \oplus L_h(E)$ , where  $L_v(E) = \ker(T\pi)$  is called the *vertical sub-bundle* and  $L_h(E)$  is the complementary *horizontal sub-bundle*. Any such splitting of  $TE$  can be regarded as an almost product structure on  $E$  for which  $L_v(E)$  and  $L_h(E)$  are its eigenbundles.

The vertical sub-bundle is defined entirely by the projection. In order to understand how the horizontal sub-bundle encodes the information about the dynamics, we will now describe the canonical lift to the tangent bundle of the configuration space.

**Definition 3.4.** The *canonical lift* on  $T\mathcal{Q}$  of a Newtonian dynamical system on  $\mathcal{Q}$  is given by the correspondence to the differential equations (3.2) of a *second order vector field*  $\Sigma \in \Gamma(T(T\mathcal{Q}))$  such that:

- (a) Integral curves of  $\Sigma$  are obtained as tangent lifts of curves on  $\mathcal{Q}$ , i.e.  $h(t) = Tg(t, 1) \in T\mathcal{Q}$  where  $h(t)$  is an integral curve of  $\Sigma$  and  $g : \mathbb{R} \rightarrow \mathcal{Q}$ .
- (b) The bundle projection  $\pi : T\mathcal{Q} \rightarrow \mathcal{Q}$  defines a trajectory  $t \mapsto \pi \circ h(t) \in \mathcal{Q}$  of the dynamical system on  $\mathcal{Q}$ .

This uniquely defines the second order vector field  $\Sigma$ . In a local chart  $(\pi^{-1}(U), q^i, v^i)$  on  $T\mathcal{Q}$  induced by a local chart  $(U, \phi) = (U, q^i)$  on  $\mathcal{Q}$ , its expression is

$$\Sigma = v^i \frac{\partial}{\partial q^i} + \Phi^i(q, v) \frac{\partial}{\partial v^i} ,$$

such that the equivalent system of first order differential equations is

$$\frac{dq^i}{dt} = v^i \quad \text{and} \quad \frac{dv^i}{dt} = \Phi^i(q, v) .$$

In this case, an equivalent statement is that  $X \in \Gamma(T(T\mathcal{Q}))$  is a vertical vector field if its action on functions which are constant along the fibers vanishes, i.e.  $\mathcal{L}_X(\pi^*f) = 0$  for all  $f \in C^\infty(\mathcal{Q})$ , where  $\mathcal{L}$  denotes the Lie derivative. Using the identity  $\mathcal{L}_{[X,Y]} = \mathcal{L}_X \mathcal{L}_Y - \mathcal{L}_Y \mathcal{L}_X$ , it follows that  $\mathcal{L}_{[X,Y]}(\pi^*f) = 0$  for all  $f \in C^\infty(\mathcal{Q})$  if  $X, Y \in \Gamma(L_v)$  are vertical vector fields. Hence  $[X, Y] \in$

$\Gamma(L_v(TQ))$  and  $L_v(TQ)$  is an involutive distribution. Thus it is Frobenius integrable, and so it describes the foliation of  $TQ$  with the fibers as leaves.

On the other hand, the *vertical lift*  $X_v \in \Gamma(L_v(TQ))$  of a vector field  $X \in \Gamma(TQ)$  is the infinitesimal generator of translations along the fibers, i.e. the one-parameter group of diffeomorphisms defined by  $\mathbb{R} \ni t \mapsto (q, tX|_q) \in TQ$ . This defines a map  $\rho : \Gamma(TQ) \rightarrow \Gamma(L_v(TQ))$  which in local coordinates reads

$$\rho : X = X^i \frac{\partial}{\partial q^i} \mapsto X_v = (\pi^* X^i) \frac{\partial}{\partial v^i} ,$$

where the components  $\pi^* X^i$  are functions which are constant along the fibers. Thus  $\{\frac{\partial}{\partial v^i}\}$  locally spans  $\Gamma(L_v(TQ))$ , and so  $[X_v, Y_v] = 0$  for all  $X_v, Y_v \in \Gamma(L_v(TQ))$ .

In order to describe the horizontal distribution induced by  $\Sigma$  on  $T(TQ)$ , we need to introduce the vertical endomorphism of  $T(TQ)$ .

**Definition 3.5.** The *vertical endomorphism*  $S \in \text{End}(T(TQ))$  is the  $(1,1)$ -tensor field which is the composition of the vertical lift and the tangent projection:  $S = \rho \circ T\pi$ , or equivalently the endomorphism of  $T(TQ)$  which makes the diagram

$$\begin{array}{ccc} T(TQ) & \xrightarrow{T\pi} & TQ \\ & \searrow S & \downarrow \rho \\ & & T(TQ) \end{array}$$

commute.

The tensor  $S$  is called the vertical endomorphism because when acting on vector fields,  $\ker(S) = \text{im}(S) = \Gamma(L_v(TQ))$  and  $S^2 = 0$ . This also implies that  $S$  is integrable, i.e. it has vanishing Nijenhuis tensor  $N_S = 0$ , so that  $S$  defines a nilpotent structure, and that in local coordinates it is given by

$$S = \frac{\partial}{\partial v^i} \otimes dq^i .$$

It can be shown [46] that  $(\mathcal{L}_\Sigma S)^2 = \mathbf{1}$ . It is also shown in [46] that  $L_+ = L_v(TQ)$  is the  $+1$ -eigenbundle of  $\mathcal{L}_\Sigma S$ . The horizontal sub-bundle  $L_- = L_h(TQ)$  is therefore the  $-1$ -eigenbundle of  $\mathcal{L}_\Sigma S$  and its elements, as *horizontal lifts* of vector fields  $X \in \Gamma(TQ)$ , take the form  $X_h = \frac{1}{2}([X_v, \Sigma] + X^c)$ , where  $X^c$  is the complete lift<sup>7</sup> of  $X$ . The rank of  $L_v(TQ)$  is  $d$ , and since  $L_h(TQ)$  is the complementary sub-bundle of  $L_v(TQ)$  it also has rank  $d$ . Thus the second order vector field  $\Sigma$ , together with the naturally defined maps  $\rho$  and  $T\pi$ , define an almost para-complex structure  $K = \mathcal{L}_\Sigma S$  on  $M = TQ$ . As we have seen, the  $+1$ -eigenbundle  $L_+ = L_v(TQ)$  is Frobenius integrable and the corresponding foliation  $\mathcal{F}_+$  is canonically identified with the fibers of  $TQ$ , i.e. the space of velocities  $v$ .

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<sup>7</sup>Let  $X \in \Gamma(TQ)$  and let  $\varphi_t$  be the local one-parameter group of diffeomorphisms on  $Q$  generated by  $X$ . The infinitesimal generator  $X^c \in \Gamma(T(TQ))$  of the local one-parameter group of diffeomorphisms on  $TQ$  defined by the canonical lift  $\varphi_t^c = T\varphi_t$  is called the *complete lift* of  $X$ . It defines a canonical injection  $\Gamma(TQ) \rightarrow \Gamma(T(TQ))$  by the directional derivatives  $X^c(\pi^* f) = \pi^* X(f)$  for  $f \in C^\infty(Q)$ . In local coordinates, if  $X = X^i \frac{\partial}{\partial q^i}$  then

$$X^c = X^i \frac{\partial}{\partial q^i} + v^i \frac{\partial X^j}{\partial q^i} \frac{\partial}{\partial v^j} .$$

In local coordinates a horizontal lift reads

$$X_{\mathbf{h}} = (\pi^* X^i) D_i \quad \text{with} \quad D_i = \left( \frac{\partial}{\partial q^i} \right)_{\mathbf{h}} = \frac{\partial}{\partial q^i} + \frac{1}{2} \frac{\partial \Phi^k}{\partial v^i} \frac{\partial}{\partial v^k},$$

where  $\{D_i\}$  is a local basis spanning  $\Gamma(L_{\mathbf{h}}(T\mathcal{Q}))$ . This easily shows that  $L_{\mathbf{h}}(T\mathcal{Q})$  is not an integrable distribution, as

$$[D_i, D_j] = \frac{1}{2} \left( \frac{\partial^2 \Phi^k}{\partial q^i \partial v^j} - \frac{\partial^2 \Phi^k}{\partial v^i \partial q^j} + \frac{1}{2} \left( \frac{\partial^2 \Phi^k}{\partial v^j \partial v^m} \frac{\partial \Phi^m}{\partial v^i} - \frac{\partial^2 \Phi^k}{\partial v^i \partial v^m} \frac{\partial \Phi^m}{\partial v^j} \right) \right) \frac{\partial}{\partial v^k},$$

so that  $[D_i, D_j] \in \Gamma(L_{\mathbf{v}}(T\mathcal{Q}))$ .

We can now obtain the local expressions of the 1-forms  $\tau^i$  and  $\alpha^i$  dual to the spans of  $\Gamma(L_{\mathbf{v}}(T\mathcal{Q}))$  and  $\Gamma(L_{\mathbf{h}}(T\mathcal{Q}))$ . Imposing the duality conditions

$$\tau^i \left( \frac{\partial}{\partial v^j} \right) = \alpha^i(D_j) = \delta^i_j \quad \text{and} \quad \tau^i(D_j) = \alpha^i \left( \frac{\partial}{\partial v^j} \right) = 0,$$

we obtain

$$\alpha^i = dq^i \quad \text{and} \quad \tau^i = dv^i - \frac{1}{2} \frac{\partial \Phi^i}{\partial v^j} dq^j.$$

Hence the local expression of the dynamical almost para-complex structure is

$$K = \mathcal{L}_{\Sigma} S = \frac{\partial}{\partial v^i} \otimes \tau^i - D_i \otimes dq^i.$$

Similarly the projections  $\Pi_{\pm} = \frac{1}{2} (1 \pm \mathcal{L}_{\Sigma} S)$  are locally given by

$$\Pi_+ = \frac{\partial}{\partial v^i} \otimes \tau^i \quad \text{and} \quad \Pi_- = D_i \otimes dq^i.$$

Finally it is straightforward to compute that the Nijenhuis tensor associated to  $K = \mathcal{L}_{\Sigma} S$  is

$$N_K = 2 [D_i, D_j] \otimes dq^i \otimes dq^j,$$

showing once more that the complete integrability of the almost para-complex structure is violated by the horizontal eigenbundle.

**Remark 3.6.** This construction is reminiscent of the definition of a linear connection on principal and vector bundles, which also introduces a splitting of the tangent bundle of the total space. However the condition here defining the horizontal sub-bundle is different and generally weaker than the usual condition for connections on an associated vector bundle. Because of this such a construction is called a *non-linear connection*. The properties of non-linear connections on the tangent bundle are discussed in [47, 48].

### 3.2 Lagrangian Dynamics and Born Geometry

Our aim now is to connect the completely general discussion above to a specific case, the description of Newtonian dynamical systems admitting a regular Lagrangian. In this framework we will encounter a simple instance of para-Kähler geometry.

We first review some useful notions about the Lagrangian formalism.



**Definition 3.7.** Let  $\mathcal{L} \in C^\infty(T\mathcal{Q})$  be a smooth function. The 1-form  $\theta_{\mathcal{L}} = S(d\mathcal{L})$  is the *Cartan 1-form* associated with  $\mathcal{L}$ . The closed 2-form  $\Omega_{\mathcal{L}} = -d\theta_{\mathcal{L}}$  is the *Lagrangian 2-form*. The function  $\mathcal{L}$  is a *regular Lagrangian* if and only if  $\Omega_{\mathcal{L}}$  is non-degenerate, and hence a symplectic form.

From this definition it follows that the Cartan 1-form is horizontal: If  $X \in \Gamma(T(T\mathcal{Q}))$ , then  $S(X) \in \Gamma(L_{\mathbf{v}}(T\mathcal{Q}))$  is a vertical vector field, so  $\theta_{\mathcal{L}}(S(X)) = S(d\mathcal{L})(S(X)) = d\mathcal{L}(S^2(X)) = 0$ .

The Euler-Lagrange equations read as

$$\mathcal{L}_{\Sigma}\theta_{\mathcal{L}} - d\mathcal{L} = 0 .$$

By applying the Cartan formula

$$\mathcal{L}_{\Sigma} = d\iota_{\Sigma} + \iota_{\Sigma}d$$

for the Lie derivative we obtain

$$\iota_{\Sigma}\Omega_{\mathcal{L}} = d\mathcal{E}_{\mathcal{L}} ,$$

where  $\mathcal{E}_{\mathcal{L}} = \iota_{\Sigma}\theta_{\mathcal{L}} - \mathcal{L}$ . This shows that  $\Sigma$  is the Hamiltonian vector field of the Hamiltonian function  $\mathcal{E}_{\mathcal{L}}$ , which is globally defined on  $T\mathcal{Q}$  because it is directly derived from the Lagrangian function.

In local coordinates  $(q^i, v^i)$  on  $T\mathcal{Q}$ , the Cartan 1-form reads

$$\theta_{\mathcal{L}} = \frac{\partial \mathcal{L}}{\partial v^i} dq^i ,$$

so that the Hamiltonian function is given by

$$\mathcal{E}_{\mathcal{L}} = v^i \frac{\partial \mathcal{L}}{\partial v^i} - \mathcal{L} ,$$

and the Lagrangian 2-form is

$$\Omega_{\mathcal{L}} = \frac{\partial^2 \mathcal{L}}{\partial v^i \partial v^j} dq^i \wedge dv^j + \frac{1}{2} \left( \frac{\partial^2 \mathcal{L}}{\partial v^i \partial q^j} - \frac{\partial^2 \mathcal{L}}{\partial q^i \partial v^j} \right) dq^i \wedge dq^j .$$

The local expression of  $\theta_{\mathcal{L}}$  explicitly shows that it is a horizontal 1-form, while the local form of  $\Omega_{\mathcal{L}}$  gives another formulation of the regularity requirement for the Lagrangian:  $\Omega_{\mathcal{L}}$  is non-degenerate if  $\ker(\Omega_{\mathcal{L}}) = \{X \in \Gamma(T(T\mathcal{Q})) : \iota_X \Omega_{\mathcal{L}} = 0\} = 0$ . Hence, given any vector field  $X = X^i \frac{\partial}{\partial q^i} + \tilde{X}^i \frac{\partial}{\partial v^i} \in \Gamma(T(T\mathcal{Q}))$ , we compute

$$\iota_X \Omega_{\mathcal{L}} = \frac{\partial^2 \mathcal{L}}{\partial v^i \partial v^j} X^i dv^j + \left( \left( \frac{\partial^2 \mathcal{L}}{\partial v^i \partial q^j} - \frac{\partial^2 \mathcal{L}}{\partial q^i \partial v^j} \right) X^i - \frac{\partial^2 \mathcal{L}}{\partial v^i \partial v^j} \tilde{X}^i \right) dq^j .$$

This shows that  $\iota_X \Omega_{\mathcal{L}} \neq 0$  for any  $X \neq 0$  if and only if  $\det \left( \frac{\partial^2 \mathcal{L}}{\partial v^i \partial v^j} \right) \neq 0$ . Thus a Lagrangian  $\mathcal{L}$  is regular if and only if the Hessian matrix  $\left( \frac{\partial^2 \mathcal{L}}{\partial v^i \partial v^j} \right)$  has maximum rank  $d$ .

It can also be shown [46] that

$$\Omega_{\mathcal{L}}(X_{\mathbf{v}}, Y_{\mathbf{v}}) = \Omega_{\mathcal{L}}(X_{\mathbf{h}}, Y_{\mathbf{h}}) = 0 \quad \text{and} \quad \Omega_{\mathcal{L}}(X_{\mathbf{v}}, Y_{\mathbf{h}}) + \Omega_{\mathcal{L}}(X_{\mathbf{h}}, Y_{\mathbf{v}}) = 0 , \quad (3.8)$$

for all  $X_{\mathbf{v}}, Y_{\mathbf{v}} \in \Gamma(L_{\mathbf{v}}(T\mathcal{Q}))$  and  $X_{\mathbf{h}}, Y_{\mathbf{h}} \in \Gamma(L_{\mathbf{h}}(T\mathcal{Q}))$ . This implies that the local expression of the

Lagrangian 2-form can be written as

$$\Omega_{\mathcal{L}} = \eta_{ij} dq^i \wedge \tau^j ,$$

where  $\eta_{ij} = \Omega_{\mathcal{L}}(D_i, \frac{\partial}{\partial v^j}) = \eta_{ji}$ . The vanishing conditions (3.8) give the compatibility between the almost para-complex structure  $K = \mathcal{L}_{\Sigma}S$  and the Lagrangian 2-form.

**Proposition 3.9.** Let  $\mathcal{L}$  be a regular Lagrangian on  $TQ$ ,  $\Omega_{\mathcal{L}}$  the associated Lagrangian 2-form and  $K = \mathcal{L}_{\Sigma}S$  the dynamical almost para-complex structure on  $TQ$ . Then the  $(0, 2)$ -tensor field  $\eta_{\mathcal{L}}$  defined by

$$\eta_{\mathcal{L}}(X, Y) = \Omega_{\mathcal{L}}(K(X), Y)$$

for  $X, Y \in \Gamma(T(TQ))$  is a metric tensor with Lorentzian signature  $(d, d)$ , i.e. the vertical and horizontal distributions  $L_v(TQ)$  and  $L_h(TQ)$  are maximally isotropic with respect to  $\eta_{\mathcal{L}}$ . Thus  $(TQ, K, \eta_{\mathcal{L}})$  is an  $L_v(TQ)$ -para-Kähler manifold with fundamental 2-form  $\Omega_{\mathcal{L}}$ .

*Proof.* The non-degeneracy of  $\eta_{\mathcal{L}}$  follows from the non-degeneracy of both  $K$  and  $\Omega_{\mathcal{L}}$ . Recall that  $K(X_v) = X_v$  and  $K(X_h) = -X_h$ . It follows that

$$\begin{aligned} \eta_{\mathcal{L}}(X_v, Y_v) &= \Omega_{\mathcal{L}}(K(X_v), Y_v) = \Omega_{\mathcal{L}}(X_v, Y_v) = 0 , \\ \eta_{\mathcal{L}}(X_h, Y_h) &= \Omega_{\mathcal{L}}(K(X_h), Y_h) = -\Omega_{\mathcal{L}}(X_h, Y_h) = 0 , \\ \eta_{\mathcal{L}}(X_v, Y_h) &= \Omega_{\mathcal{L}}(K(X_v), Y_h) = \Omega_{\mathcal{L}}(X_v, Y_h) = \eta_{ij} X_v^i Y_h^j , \end{aligned}$$

in which we used (3.8) and the local expression of  $\Omega_{\mathcal{L}}$ . From the last equation, we infer that  $\eta_{\mathcal{L}}$  is symmetric, since  $\eta_{ij}$  is symmetric. Therefore  $\eta_{\mathcal{L}}$  defines a metric compatible with  $K$  and the Lagrangian 2-form. It also follows that the two eigenbundles  $L_v(TQ)$  and  $L_h(TQ)$  of  $K$  are isotropic with respect to  $\eta_{\mathcal{L}}$  and, since they are both of rank  $d$ , they are maximal. We have already seen that only  $L_v(TQ)$  is an integrable eigenbundle. Thus  $(TQ, K = \mathcal{L}_{\Sigma}S, \eta_{\mathcal{L}})$  is an  $L_v(TQ)$ -para-Kähler manifold, since  $\Omega_{\mathcal{L}}$  is symplectic for a regular Lagrangian.  $\square$

In local coordinates, the metric  $\eta_{\mathcal{L}}$  takes the form

$$\eta_{\mathcal{L}} = \eta_{ij} (dq^i \otimes \tau^j + \tau^i \otimes dq^j) \quad \text{with} \quad \eta_{ij} = \frac{\partial^2 \mathcal{L}}{\partial v^i \partial v^j} ,$$

showing once more the importance of the regularity condition for the Lagrangian.

Any dynamical system described by a regular Lagrangian  $\mathcal{L}$  induces an almost Kähler structure on the tangent bundle  $TQ$  (see e.g. [46]). The almost complex structure  $I$  on the tangent bundle, associated with the second order vector field  $\Sigma \in \Gamma(T(TQ))$ , is given by

$$I = S + \frac{1}{2} K \Pi_+ ,$$

where  $K = \mathcal{L}_{\Sigma}S$  is the almost para-complex structure associated with  $\Sigma$  and  $\Pi_+ = \frac{1}{2}(\mathbb{1} + K)$  its vertical projector. It is easy to show that  $I(X_v) = -X_h$  and  $I(X_h) = X_v$ , where  $X_h$  and  $X_v$  are, respectively, the horizontal and vertical lift of a vector field  $X \in \Gamma(TQ)$ . In local coordinates, by

fixing the splitting  $T(T\mathcal{Q}) = L_v(T\mathcal{Q}) \oplus L_h(T\mathcal{Q})$  the almost complex structure  $I$  reads

$$I = \frac{\partial}{\partial v^i} \otimes dq^i - D_i \otimes \tau^i .$$

Given the properties of the Lagrangian 2-form  $\Omega_{\mathcal{L}}$ , it follows that it is compatible with the almost complex structure  $I$ , i.e. they satisfy the relation

$$\Omega_{\mathcal{L}}(I(X), Y) + \Omega_{\mathcal{L}}(X, I(Y)) = 0 ,$$

for all  $X, Y \in \Gamma(T(T\mathcal{Q}))$ . We can then introduce the Hermitian metric

$$\mathcal{H}_{\mathcal{L}}(X, Y) = \Omega_{\mathcal{L}}(I(X), Y) ,$$

such that

$$\mathcal{H}_{\mathcal{L}}(I(X), I(Y)) = \mathcal{H}_{\mathcal{L}}(X, Y) ,$$

for all  $X, Y \in \Gamma(T(T\mathcal{Q}))$ . In local coordinates, it has the expression

$$\mathcal{H}_{\mathcal{L}} = \eta_{ij} (dq^i \otimes dq^j + \tau^i \otimes \tau^j) ,$$

so that for a regular Lagrangian it defines a Riemannian metric on  $T\mathcal{Q}$ .

It follows that the Lagrangian almost para-Kähler structure and almost Kähler structure have the same fundamental 2-form. With this data it is then straightforward to show that the almost para-Kähler structure<sup>8</sup>  $(K, \eta_{\mathcal{L}}, \Omega_{\mathcal{L}})$  and the almost Kähler structure  $(I, \mathcal{H}_{\mathcal{L}}, \Omega_{\mathcal{L}})$  on  $T\mathcal{Q}$  satisfy the relations

$$\eta_{\mathcal{L}}^{-1} \mathcal{H}_{\mathcal{L}} = \mathcal{H}_{\mathcal{L}}^{-1} \eta_{\mathcal{L}} \quad \text{and} \quad \Omega_{\mathcal{L}}^{-1} \mathcal{H}_{\mathcal{L}} = -\mathcal{H}_{\mathcal{L}}^{-1} \Omega_{\mathcal{L}} .$$

Thus  $(\eta_{\mathcal{L}}, \Omega_{\mathcal{L}}, \mathcal{H}_{\mathcal{L}})$  is a Born geometry and the tangent bundle  $T\mathcal{Q}$  of the configuration space  $\mathcal{Q}$ , for a dynamical system arising from a regular Lagrangian, is a Born manifold. The chiral structure  $(\eta_{\mathcal{L}}, J_{\mathcal{L}})$ , introduced in the usual way by  $J_{\mathcal{L}} = \eta_{\mathcal{L}}^{-1} \mathcal{H}_{\mathcal{L}}$ , makes the triple  $(I, J_{\mathcal{L}}, K)$  an *almost para-quaternionic structure* [17, 49]. Generalizing this construction, we can infer that any almost para-Hermitian structure and almost Hermitian structure having the same fundamental 2-form and the same splitting of the tangent bundle give rise to a Born geometry.

**Example 3.10.** A particular instance of this geometry [46] is given by geodesic motion on any Riemannian manifold  $\mathcal{Q}$  with metric tensor  $g = g_{ij} dq^i \otimes dq^j$ . In this case the Lagrangian function is given by  $\mathcal{L} = \frac{1}{2} g_{ij}(q) v^i v^j$  and the second order vector field is

$$\Sigma = v^i \frac{\partial}{\partial q^i} - \Gamma_{km}^i(q) v^k v^m \frac{\partial}{\partial v^i} ,$$

where  $\Gamma_{km}^i$  are the Christoffel symbols of the Levi-Civita connection compatible with  $g$ . The eigenbundle  $L_h(T\mathcal{Q})$  here is the horizontal distribution of the Levi-Civita connection and has a local frame given by the vector fields

$$D_i = \frac{\partial}{\partial q^i} - \Gamma_{ij}^k(q) v^j \frac{\partial}{\partial v^k} \quad \text{with} \quad [D_i, D_j] = R^k_{ijm}(q) v^m \frac{\partial}{\partial v^k} ,$$

---

<sup>8</sup>It is redundant here to write all three of the tensors, but we wish to stress what objects are involved in the structures under consideration.

where  $R^k_{ijm}$  are the components of the Riemann curvature tensor of  $g$ ; in other words, the horizontal distribution is locally spanned by the tangent vectors of the paths in  $T\mathcal{Q}$  defined via parallel transport of a vector  $v \in T_q\mathcal{Q}$  along paths through  $q$  in  $\mathcal{Q}$ . Thus  $(T\mathcal{Q}, K, \eta_{\mathcal{L}})$  is a para-Kähler manifold if and only if  $g$  is a flat metric. In general  $L_{\mathbf{v}}(T\mathcal{Q})$  is, as always, canonically identified with the tangent bundle of the space of velocities, while here the inverse metric  $g^{-1} = g^{ij} \frac{\partial}{\partial q^i} \otimes \frac{\partial}{\partial q^j}$  identifies  $L_{\mathbf{h}}(T\mathcal{Q})$  with the cotangent bundle. The fundamental Lagrangian 2-form is

$$\Omega_{\mathcal{L}} = g_{ij}(q) dq^i \wedge \tau^j \quad \text{with} \quad \tau^i = dv^i + \Gamma^i_{jk}(q) v^k dq^j ,$$

while the generalized metric is the Sasaki metric on  $T\mathcal{Q}$  [15]:

$$\mathcal{H}_{\mathcal{L}} = g_{ij}(q) (dq^i \otimes dq^j + \tau^i \otimes \tau^j) ,$$

This is similar to the example presented in [16], in which an arbitrary connection compatible with  $g$  is used in the definition of the horizontal sub-bundle, with the difference that in our case  $K$  is always an almost para-Kähler structure for any choice of metric  $g$ .

It also follows generally that the D-bracket and the bracket associated to the Levi-Civita connection compatible with  $\eta_{\mathcal{L}}$  coincide, since  $(T\mathcal{Q}, K, \eta_{\mathcal{L}})$  is almost para-Kähler.

### 3.3 Para-Kähler Geometry of Phase Spaces

We shall now describe how the Legendre transform of a Lagrangian function allows one to import the para-Kähler structure from the tangent bundle  $T\mathcal{Q}$  to the cotangent bundle  $T^*\mathcal{Q}$  of a configuration space  $\mathcal{Q}$ . Assuming the existence of a regular Lagrangian also implies that the Legendre transform is well-defined as the fiber derivative of the Lagrangian function  $\mathcal{L}$ , hence all the structures defined thus far can also be introduced on the cotangent bundle  $T^*\mathcal{Q}$ . Let us recall the definition of Legendre transform [50], for a more general statement see [51].

**Definition 3.11.** The *Legendre transform* of a Lagrangian function  $\mathcal{L} \in C^\infty(T\mathcal{Q})$  is the fiber derivative

$$F\mathcal{L} : T\mathcal{Q} \longrightarrow T^*\mathcal{Q}$$

given, at any point  $q \in \mathcal{Q}$ , by

$$(F\mathcal{L}(v))(z) = \left. \frac{d}{dt} \mathcal{L}(q, v + tz) \right|_{t=0} \quad \text{with} \quad F\mathcal{L}(v) \in T^*_q\mathcal{Q} ,$$

for all  $v, z \in T_q\mathcal{Q}$ . This transformation is fiber preserving, and in a local chart it reads

$$F\mathcal{L} : (q^i, v^i) \longmapsto (q^i, p_i) \quad \text{with} \quad p_i := (F\mathcal{L}(v))_i = \frac{\partial \mathcal{L}}{\partial v^i} .$$

If  $\mathcal{L}$  is a regular Lagrangian, then the Legendre transform defines a local diffeomorphism between  $T\mathcal{Q}$  and  $T^*\mathcal{Q}$ . A regular Lagrangian for which this diffeomorphism is globally defined will be called *hyper-regular*. From now on, we will assume hyper-regularity of the Lagrangian.

The pushforward  $F\mathcal{L}_* : \Gamma(T(T\mathcal{Q})) \rightarrow \Gamma(T(T^*\mathcal{Q}))$  in local coordinates, for any vector field  $X =$

$X^i \frac{\partial}{\partial q^i} + \tilde{X}^i \frac{\partial}{\partial v^i} \in \Gamma(T(T\mathcal{Q}))$ , is given by

$$X \mapsto X_{\mathcal{L}} = X^i \frac{\partial}{\partial q^i} + \mathcal{L}_X \left( \frac{\partial \mathcal{L}}{\partial v^i} \right) \Big|_{(q,p)} \frac{\partial}{\partial p_i} .$$

Using this definition, it is easy to check that there exists a unique 1-form  $\theta_0 \in \Omega^1(T^*\mathcal{Q})$  such that  $(F\mathcal{L})^*\theta_0 = \theta_{\mathcal{L}}$ , where  $\theta_{\mathcal{L}}$  is the Cartan 1-form on  $T\mathcal{Q}$ . In local coordinates it is given by  $\theta_0 = p_i dq^i$ . A similar statement holds for the Lagrangian 2-form  $\Omega_{\mathcal{L}}$ : There exists a unique closed 2-form  $\omega_0$  on  $T^*\mathcal{Q}$  that pulls back to  $\Omega_{\mathcal{L}}$ , which is just the canonical 2-form given locally in Darboux coordinates by

$$\omega_0 = dp_i \wedge dq^i .$$

Having defined  $F\mathcal{L}_*$ , we can also show how the splitting of  $T(T\mathcal{Q})$  pushes forward to a splitting of  $T(T^*\mathcal{Q})$ : The basis vectors  $\frac{\partial}{\partial v^i}$  locally spanning  $\Gamma(L_v(T\mathcal{Q}))$  push forward to  $V_i = \frac{\partial^2 \mathcal{L}}{\partial v^i \partial v^j} \Big|_{(q,p)} \frac{\partial}{\partial p_j}$ . Since  $\mathcal{L}$  is regular, the matrix  $\left( \frac{\partial^2 \mathcal{L}}{\partial v^i \partial v^j} \Big|_{(q,p)} \right)$  acts as a  $\text{GL}(d, \mathbb{R})$ -transformation of the vertical distribution, hence the basis spanning the vertical sub-bundle of  $T(T^*\mathcal{Q})$  can be written as  $Q^i = \frac{\partial}{\partial p_i}$ ; we will see that using this transformed basis does not change the almost para-complex structure  $K$  on  $T^*\mathcal{Q}$ , as expected. Thus  $F\mathcal{L}$  preserves verticality. Similarly, the horizontal distribution on  $T(T\mathcal{Q})$  pushes forward to the horizontal distribution on  $T(T^*\mathcal{Q})$  spanned by

$$H_i = \frac{\partial}{\partial q^i} + \mathcal{L}_{D_i} \left( \frac{\partial \mathcal{L}}{\partial v^j} \right) \Big|_{(q,p)} \frac{\partial}{\partial p_j} =: P_i + N_{ij} Q^j ,$$

which defines a non-linear connection. It is shown in [52] that the connection coefficients  $N_{ij}$  are symmetric for a hyper-regular Lagrangian. This will play a crucial role for the interpretation given in the following.

We can also write down the corresponding dual 1-forms

$$\zeta_i = dp_i - N_{ji} dq^j \quad \text{and} \quad dq^i ,$$

so that the almost para-complex structure induced by the splitting on  $T(T^*\mathcal{Q})$  is

$$K_N = Q^i \otimes \zeta_i - H_i \otimes dq^i = Q^i \otimes dp_i - P_i \otimes dq^i - 2 N_{ji} P^i \otimes dq^j . \quad (3.12)$$

Note that, had we considered the  $\text{GL}(d, \mathbb{R})$ -transformed vertical basis,  $K_N$  would not have changed, since the vertical dual 1-form would have changed by the inverse matrix of the transformed vertical basis.

We now need to take a step back and describe the natural para-Kähler structures of the cotangent bundle, in order to show how they are intrinsically related to this construction. We have already discussed above the natural splitting of the tangent bundle of any fiber bundle  $E \rightarrow \mathcal{Q}$  induced by its projection map. For the cotangent bundle  $T^*\mathcal{Q}$ , this implies that  $T(T^*\mathcal{Q}) = L_v^0(T^*\mathcal{Q}) \oplus L_h^C(T^*\mathcal{Q})$ , which in local Darboux coordinates is given by

$$\Gamma(L_v^0(T^*\mathcal{Q})) = \text{Span}_{C^\infty(T^*\mathcal{Q})} \{Q^i\} \quad \text{and} \quad \Gamma(L_h^C(T^*\mathcal{Q})) = \text{Span}_{C^\infty(T^*\mathcal{Q})} \{E_i = P_i + C_{ij} Q^j\} ,$$

where  $C : \Gamma(T(T^*\mathcal{Q})) \rightarrow \Gamma(L_v^0(T^*\mathcal{Q}))$  is any local map. This is the most general form that can be assigned to the horizontal distribution, since  $C$  is defined in each patch but does not necessarily

transform as a tensor. This is also called a non-linear connection, since it is obtained by the requirement that the horizontal sub-bundle is annihilated by vertical distributions, but without further assumptions,<sup>9</sup> a freedom of choice is left in the definition of a vertical 1-form that is given by all the possible choices of  $C$ . Hence the almost para-complex structure on  $T^*\mathcal{Q}$  is locally given by

$$K_C = Q^i \otimes \tau_i - E_i \otimes dq^i, \quad (3.13)$$

where  $\tau_i = dp_i - C_{ji} dq^j$  and  $dq^i$  are the dual 1-forms to  $Q^i$  and  $E_i$  respectively.

Since  $T^*\mathcal{Q}$  also has a natural symplectic structure given by the canonical 2-form  $\omega_0 = dp_i \wedge dq^i$ , we may ask if there exists a compatibility condition between  $K_C$  and  $\omega_0$  which endows  $T^*\mathcal{Q}$  with the structure of an almost para-Kähler manifold. In other words, we want to construct a metric  $\eta_C(X, Y) = \omega_0(K_C(X), Y)$  for which  $L_v^0(T^*\mathcal{Q})$  and  $L_h^C(T^*\mathcal{Q})$  are maximally isotropic distributions. We first compute the  $(0, 2)$ -tensor  $\eta_C$  by expressing  $K_C$  in Darboux coordinates to get

$$\eta_C = \omega_0 K_C = dq^i \otimes dp_i + dp_i \otimes dq^i - 2C_{ji} dq^i \otimes dq^j. \quad (3.14)$$

We notice that  $\eta_C$  is non-degenerate because  $\omega_0$  and  $K_C$  are. We can easily see that the vertical distribution  $L_v^0(T^*\mathcal{Q})$  is isotropic with respect to  $\eta_C$ :

$$\eta_C(X_v, Y_v) = \eta_C(Y_v, X_v) = 0.$$

We further have

$$\eta_C(X_h, Y_v) = \eta_C(Y_v, X_h) = (X_h)^i (Y_v)_i,$$

where  $X_h = (X_h)^i E_i \in \Gamma(L_h^C(T^*\mathcal{Q}))$  and  $Y_v = (Y_v)_i Q^i \in \Gamma(L_v^0(T^*\mathcal{Q}))$ . Requiring isotropy of the horizontal distribution, we obtain

$$\eta_C(E_i, E_j) = C_{ij} + C_{ji} - 2C_{ji} = 0.$$

Hence  $C_{ij}$  must be symmetric, or equivalently the map

$$\eta_C C : \Gamma(T(T^*\mathcal{Q})) \times \Gamma(T(T^*\mathcal{Q})) \longrightarrow C^\infty(T^*\mathcal{Q})$$

must be symmetric in each local trivialization. This condition also implies the symmetry of the tensor  $\eta_C$ , thus  $\eta_C$  is a metric for which  $L_v^0(T^*\mathcal{Q})$  and  $L_h^C(T^*\mathcal{Q})$  are maximally isotropic sub-bundles. Thus with these conditions,  $(T^*\mathcal{Q}, K_C, \eta_C)$  is an  $L_v^0(T^*\mathcal{Q})$ -para-Kähler manifold.

This result may be interpreted as follows. Any collection of locally symmetric  $(0, 2)$ -tensor fields  $C$  on  $T^*\mathcal{Q}$  corresponds to a different splitting of  $T(T^*\mathcal{Q})$  and a different metric  $\eta_C$ , which together give an almost para-Kähler structure for which the canonical 2-form  $\omega_0$  on  $T^*\mathcal{Q}$  is the fundamental 2-form. We say that such an almost para-Kähler structure on  $T^*\mathcal{Q}$  is  $\omega_0$ -compatible. The  $\omega_0$ -compatible para-Kähler structure corresponding to  $C = 0$  is called *canonical*, since it has the Levi-Civita connection as its canonical para-Hermitian connection.<sup>10</sup> In this case both  $L_v^0(T^*\mathcal{Q})$  and  $L_h^0(T^*\mathcal{Q})$ , given respectively by  $\Gamma(L_v^0(T^*\mathcal{Q})) = \text{Span}_{C^\infty(T^*\mathcal{Q})}\{Q^i\}$  and  $\Gamma(L_h^0(T^*\mathcal{Q})) = \text{Span}_{C^\infty(T^*\mathcal{Q})}\{P_i\}$ , are integrable distributions and Lagrangian with respect to  $\omega_0$ . The para-complex structure is locally given by  $K_0 = Q^i \otimes dp_i - P_i \otimes dq^i$  and  $\eta_0 = dq^i \otimes dp_i + dp_i \otimes dq^i$  is a flat metric. The deformation of

<sup>9</sup>For instance, we may assume that the vertical 1-forms transform in a prescribed way under the adjoint action of a subgroup of  $\text{GL}(d, \mathbb{R})$ , which is another way to define a linear connection.

<sup>10</sup>We consider the canonical splitting here up to constant  $C$ .

the canonical splitting by  $C : \Gamma(L_{\mathfrak{h}}^0) \rightarrow \Gamma(L_{\mathfrak{v}}^0)$  is somewhat analogous to the deformations of almost para-Hermitian structures that we discussed in Section 2.3, with the important difference that here the deformation is not realized by an  $O(d, d)(T^*\mathcal{Q})$ -transformation.

Going back to the splitting induced by a dynamical system, we see that the almost para-complex structure  $K_N$  from (3.12) is compatible with  $\omega_0$  and together they define an almost para-Kähler structure on  $T^*\mathcal{Q}$ ; in this case we identify  $C_{ij} = N_{ij}$ . Let us also stress that a bijective Legendre transform maps a Lagrangian almost para-Kähler structure on  $T\mathcal{Q}$  into one of the members of the class of  $\omega_0$ -compatible almost para-Kähler structures on  $T^*\mathcal{Q}$ , and the generalized metric  $\mathcal{H}_{\mathcal{L}}$  on  $T\mathcal{Q}$  to a generalized metric  $\mathcal{H}_C$  on  $T^*\mathcal{Q}$ .

**Example 3.15.** As in the setting of Example 3.10, consider the case that  $(\mathcal{Q}, g)$  is a Riemannian manifold of dimension  $d$  with the hyper-regular Lagrangian function  $\mathcal{L} = \frac{1}{2} g_{ij} v^i v^j$ . In this case the connection coefficients

$$C_{ij} = N_{ij} = \Gamma_{ij}^k(q) p_k$$

coincide with the Christoffel symbols and again the sub-bundle  $L_{\mathfrak{h}}^C(T^*\mathcal{Q})$  is the horizontal distribution of the Levi-Civita connection of  $g$ . As previously  $(T^*\mathcal{Q}, K_C, \eta_C)$  is a para-Kähler manifold if and only if  $g$  has vanishing curvature. A generalized metric on  $T^*\mathcal{Q}$  is defined in the generic case by the Sasaki metric [26]

$$\mathcal{H}_C = g_{ij}(q) dq^i \otimes dq^j + g^{ij}(q) \tau_i \otimes \tau_j \quad \text{with} \quad \tau_i = dp_i - \Gamma_{ij}^k(q) p_k dq^j .$$

This is a special case of the 6-parameter family of natural almost para-Hermitian structures constructed in [27] as the general natural lifts of the metric  $g$  from the configuration manifold  $\mathcal{Q}$  to the total space  $T^*\mathcal{Q}$  of its cotangent bundle.

## 4 Dynamical Nonassociativity and Generalized Fluxes

In this section we will describe some examples of how the description of dynamical systems in terms of para-Hermitian geometry goes beyond systems admitting Lagrangian functions which are regular; in these settings the almost para-Kähler structures are relaxed to almost para-Hermitian structures. In particular, we look at certain dynamical systems which do not even admit a Lagrangian function due to the presence of fluxes which induce a nonassociative deformation of the phase space Poisson algebra. This will pave the way, within a physically elementary setting, to a general understanding of how to incorporate geometric and non-geometric fluxes as deformations of the local para-Kähler geometry of the phase space  $T^*\mathcal{Q}$  of any configuration space  $\mathcal{Q}$ .

### 4.1 Para-Hermitian Geometry of a Non-Lagrangian System

We consider a particular dynamical system in which non-regularity means the lack of a globally defined (regular) Lagrangian or Hamiltonian function. Our goal is to demonstrate how a para-Hermitian structure can be introduced on the cotangent bundle in order to compensate the lack of a regular Hamiltonian, and obtain a geometric description of the dynamics of this system. The configuration space is  $\mathcal{Q} = \mathbb{R}^d$  with  $d \geq 3$  and the equations of motion are given by

$$\frac{d^2 q^i}{dt^2} = 2 \delta^{ik} B_{kj} \frac{dq^j}{dt} , \tag{4.1}$$

where  $B = \frac{1}{2} B_{ij}(q) dq^i \wedge dq^j$  is any 2-form on  $\mathcal{Q}$  (regarded as a skew-symmetric map). For  $d = 3$  this is the Lorentz force law describing the dynamics of a classical spinless point particle with unit mass and electric charge moving in a magnetic field  $\mathcal{B}^i = \varepsilon^{ijk} B_{jk}$ , where the force exerted by the electric field of the charged particle is neglected; our main interest is the case where  $\mathcal{B}$  is generated by a smooth distribution of magnetic monopoles (see e.g. [28, 29, 53–58] for other treatments of this dynamical system). In the following we work in arbitrary dimensionality since our later considerations will be naturally adapted to this general setting, but the reader interested in concrete examples may wish to keep this special case in mind.

By lifting the dynamics on the cotangent bundle  $T^*\mathcal{Q}$ , we obtain the second order vector field

$$\Sigma = \delta^{ij} p_j \frac{\partial}{\partial q^i} + 2 \delta^{jk} B_{ij} p_k \frac{\partial}{\partial p_i} \in \Gamma(T(T^*\mathcal{Q})) .$$

The corresponding integral curves yield the system of first order differential equations

$$\frac{dq^i}{dt} = \delta^{ij} p_j \quad \text{and} \quad \frac{dp_i}{dt} = 2 B_{ij} \delta^{jk} p_k ,$$

which is equivalent to (4.1). A similar expression is obtained for the lift on the tangent bundle  $T\mathcal{Q}$ . However, such dynamical systems do not generally admit any (global) Lagrangian or Hamiltonian function, hence a fiber derivative connecting the two lifts cannot be defined as previously.

The splitting  $T(T^*\mathcal{Q}) = L_v^0(T^*\mathcal{Q}) \oplus L_h^B(T^*\mathcal{Q})$  induced by  $\Sigma$  is given by

$$\begin{aligned} \Gamma(L_v^0(T^*\mathcal{Q})) &= \text{Span}_{C^\infty(T^*\mathcal{Q})} \left\{ Q^i = \frac{\partial}{\partial p_i} \right\} , \\ \Gamma(L_h^B(T^*\mathcal{Q})) &= \text{Span}_{C^\infty(T^*\mathcal{Q})} \{ D_i = P_i + B_{ij} Q^j \} , \end{aligned}$$

with  $P_i = \frac{\partial}{\partial q^i}$ . The Lie algebra defining these distributions is

$$[D_i, D_j] = (\partial_i B_{jk} - \partial_j B_{ik}) Q^k , \quad [D_i, Q^j] = 0 \quad \text{and} \quad [Q^i, Q^j] = 0 ,$$

where  $\partial_i$  denotes the partial derivative with respect to  $q^i$ . This shows that the horizontal distribution  $L_h^B(T^*\mathcal{Q})$  is not involutive, while the vertical distribution  $L_v^0(T^*\mathcal{Q})$  is integrable and can be identified with the tangent bundle of the fibers of  $T^*\mathcal{Q}$ .

The respective dual 1-forms to the basis vector fields  $Q^i$  and  $D_i$  are

$$\tilde{\theta}_i = dp_i + B_{ij} dq^j \quad \text{and} \quad dq^i ,$$

thus we can write the almost para-complex structure  $K_B$  defined by the splitting  $T(T^*\mathcal{Q}) = L_v^0(T^*\mathcal{Q}) \oplus L_h^B(T^*\mathcal{Q})$ , i.e. such that  $K_B|_{L_v^0(T^*\mathcal{Q})} = \mathbb{1}$  and  $K_B|_{L_h^B(T^*\mathcal{Q})} = -\mathbb{1}$ :

$$K_B = Q^i \otimes \tilde{\theta}_i - D_i \otimes dq^i = Q^i \otimes dp_i - P_i \otimes dq^i + 2 B_{ij} Q^i \otimes dq^j . \quad (4.2)$$

We now define a Lorentzian metric  $\eta_B$  on  $T^*\mathcal{Q}$  in order to obtain a suitable almost symplectic 2-form which gives the canonical equations of motion. The introduction of such a metric can be regarded as a way to get around the problem of the non-existence of a global Hamiltonian. The flat Lorentzian



metric with  $\eta_B(X_v, Y_h) = \eta_B(Y_h, X_v)$  and  $\eta_B(X_v, Y_v) = \eta_B(X_h, Y_h) = 0$  in local coordinates reads

$$\eta_B = dq^i \otimes \tilde{\theta}_i + \tilde{\theta}_i \otimes dq^i ,$$

and may be regarded as a lift of the natural flat Euclidean metric defined on the configuration space  $\mathcal{Q} = \mathbb{R}^d$ . Then  $(T^*\mathcal{Q}, K_B, \eta_B)$  is an  $L_v^0$ -para-Hermitian manifold. We stress that a lack of a Hamiltonian or Lagrangian function translates into a weakening of the properties of the carrier manifold  $M$ , i.e.  $T\mathcal{Q}$  (or  $T^*\mathcal{Q}$ ) endowed with a regular Lagrangian (or Hamiltonian) is an almost para-Kähler manifold, while  $T\mathcal{Q}$  (or  $T^*\mathcal{Q}$ ) without a regular function is only an almost para-Hermitian manifold.

Our main goal now is to obtain the almost symplectic 2-form describing the coordinate algebra on the phase space from the geometry of the phase space itself. In this case, such a 2-form is given by the fundamental 2-form of para-Hermitian geometry, i.e.  $\omega_B(X, Y) = \eta_B(K_B(X), Y)$ . In local coordinates  $(q^i, p_i)$  it reads

$$\omega_B = \tilde{\theta}_i \wedge dq^i = dp_i \wedge dq^i + 2 B_{ij} dq^i \wedge dq^j , \quad (4.3)$$

and its inverse leads to the coordinate algebra

$$\{q^i, q^j\}_B = 0 , \quad \{q^i, p_j\}_B = \delta^i_j \quad \text{and} \quad \{p_i, p_j\}_B = 2 B_{ij}(q) .$$

These define twisted Poisson brackets which have non-zero Jacobiators amongst the fiber momentum coordinates given by

$$\{p_i, p_j, p_k\}_B = 3 H_{ijk}(q) ,$$

where  $H = dB = \frac{1}{3!} H_{ijk}(q) dq^i \wedge dq^j \wedge dq^k$  is a 3-form on  $\mathcal{Q}$  with  $H_{ijk} = \partial_{[i} B_{jk]}$ . The nonassociativity of the coordinate algebra is related to the lack of closure of the fundamental 2-form  $\omega_B$ :

$$d\omega_B = 2 H ,$$

in which we see the emergence of  $H$ -flux.<sup>11</sup> This algebra is associative only when the  $H$ -flux vanishes; for  $d = 3$  this is the classical Maxwell theory, where  $\partial_i B^i = 0$ . In general, the dynamical vector field  $\Sigma$  is Hamiltonian with respect to  $\omega_B$  for the locally defined Hamiltonian function  $\mathcal{E} = \frac{1}{2} \delta^{ij} p_i p_j$ .

In the setting of para-Hermitian geometry, the nonassociativity of the coordinate algebra means the violation of the weak integrability condition (Definition 2.29). In order to show how locally non-geometric fluxes obstruct the relative weak integrability in this example, we compute the D-bracket on  $(T^*\mathcal{Q}, K_B, \eta_B)$  and the bracket associated to the Levi-Civita connection compatible with  $\eta_B$  (which is the D-bracket when  $B = 0$ ), and then compare them using (2.25). Here  $\eta_B$  is a flat metric, hence the Levi-Civita connection has vanishing Christoffel symbols, and we also know that  $\llbracket Q^i, Q^j \rrbracket^D = \llbracket Q^i, Q^j \rrbracket^{\nabla^{\text{LC}}} = 0$  from Proposition 2.26, i.e.  $L_v^0(T^*\mathcal{Q})$  is a weakly integrable distribution. By computing

$$\eta_B(\nabla_{D_i}^{\text{LC}} D_j - \nabla_{D_j}^{\text{LC}} D_i, Z) = Z^k (\partial_i B_{jk} - \partial_j B_{ik}) \quad \text{and} \quad \eta_B(\nabla_Z^{\text{LC}} D_i, D_j) = Z^k \partial_k B_{ij} ,$$

where  $Z = Z^i D_i + \tilde{Z}_i Q^i \in \Gamma(T(T^*\mathcal{Q}))$ , we find that the bracket associated to the Levi-Civita

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<sup>11</sup>It is natural to think of this flux as the geometric NS-NS  $H$ -flux, and later on it will indeed be identified in that way; for  $d = 3$  the  $H$ -flux can be interpreted as a field of magnetic charges in the present context.

connection for two horizontal basis elements is given by

$$\eta_B(\llbracket D_i, D_j \rrbracket^{\nabla^{\text{LC}}}, Z) = H_{ijk} Z^k .$$

Considering the canonical para-Hermitian connection, which for the almost para-complex structure  $K_B$  is not the Levi-Civita connection, the D-bracket is given by

$$\eta(\llbracket D_i, D_j \rrbracket_B^{\text{D}}, Z) = 0 ,$$

and thus  $L_{\mathbf{h}}^B(T^*\mathcal{Q})$  is also weakly integrable with respect to  $K_B$ , as it should be. The difference between the D-bracket and the  $\nabla^{\text{LC}}$ -bracket is exactly measured by  $d\omega_B = 2H$ , as formulated by (2.25).

Following [19], we may give a new perspective on this dynamical nonassociativity, based on the flux deformations of almost para-Hermitian structures that we discussed in Section 2. The almost para-Hermitian structure  $(K_B, \eta_B)$  on  $T^*\mathcal{Q}$  can be regarded as a deformation via a  $B_-$ -transformation of the canonical para-Kähler structure  $(K_0, \eta_0)$ , where  $\eta_B = \eta_0$  since by definition  $e^{B_-} \in \text{O}(d, d)(T^*\mathcal{Q})$ , whereas the closure of  $\omega_0$  is no longer preserved by  $\omega_B = \eta_B K_B$ . In the present case the map  $B_- : \Gamma(L_{\mathbf{h}}^0(T^*\mathcal{Q})) \rightarrow \Gamma(L_{\mathbf{v}}^0(T^*\mathcal{Q}))$  is defined by

$$B_- = B_{ij} Q^i \otimes dq^j$$

and it satisfies the skew condition

$$b_- = \eta_B B_- = B_{ij} dq^i \wedge dq^j ,$$

with  $b_-$  having only a  $(+0, -2)$ -component with respect to the canonical splitting<sup>12</sup>  $T(T^*\mathcal{Q}) = L_{\mathbf{v}}^0(T^*\mathcal{Q}) \oplus L_{\mathbf{h}}^0(T^*\mathcal{Q})$ . In this sense the horizontal distribution of the dynamical splitting of  $T(T^*\mathcal{Q})$  can be regarded as the graph  $\Gamma(L_{\mathbf{h}}^B(T^*\mathcal{Q})) = \text{Graph}_{T(T^*\mathcal{Q})}(B_-) = \{Z + B_-(Z) : Z \in \Gamma(T(T^*\mathcal{Q}))\}$ , using standard terminology from generalized geometry. Thus  $K_B = e^{B_-} K_0 e^{-B_-}$  in (4.2) is exactly given by (2.28), and the fundamental 2-form is given by  $\omega_B = \omega_0 + 2b_-$  as confirmed by its local form (4.3). This description also confirms that fluxes are generally a relative notion obstructing the compatibility of two (almost) para-Hermitian structures in the form of structure constants of the D-bracket algebra of vector fields.

## 4.2 Born Reciprocity and the $R$ -Flux Model

An important application of the non-Lagrangian dynamical system discussed above comes from applying the duality transformation  $(q^i, p_i) \mapsto (p_i, -q^i)$  of order 4 [28], sometimes called *Born reciprocity*. Born reciprocity is a symplectomorphism of the canonical 2-form  $\omega_0$  on  $T^*\mathcal{Q}$ , but it does not preserve the canonical para-Kähler structure because it sends  $K_0 \mapsto -K_0$  and  $\eta_0 \mapsto -\eta_0$ . In fact, this transformation can be understood as a composition of deformations of para-Hermitian structures: It sends the 2-form  $B = \frac{1}{2} B_{ij}(q) dq^i \wedge dq^j$  on the configuration space  $\mathcal{Q}$  to the 2-form  $\beta = \frac{1}{2} \beta^{ij}(p) dp_i \wedge dp_j$  on the fiber spaces of the cotangent bundle  $\pi : T^*\mathcal{Q} \rightarrow \mathcal{Q}$ , and correspondingly

<sup>12</sup>In Section 2 we considered a  $B_+$ -transformation associated to a  $(+2, -0)$ -form  $b_+$ . In the present case we consider instead a  $B_-$ -transformation which is associated to a 2-form  $b_-$  on the base manifold  $\mathcal{Q}$ , hence it becomes a  $(+0, -2)$ -form since the coordinates on the base manifold are the adapted coordinates of the horizontal eigenbundle  $L_{\mathbf{h}}^0$ , which has eigenvalue  $-1$ .

the  $H$ -flux  $H_{ijk} = \partial_{[i} B_{jk]}$  to the  $R$ -flux<sup>13</sup>  $R^{ijk} = \tilde{\partial}^{[i} \beta^{jk]}$ , where  $\tilde{\partial}^i$  denotes the partial derivative with respect to  $p_i$ . It therefore sends the map  $B_- : \Gamma(L_h^0)(T^*Q) \rightarrow \Gamma(L_v^0)(T^*Q)$  to the map  $\beta_+ : \Gamma(L_v^0)(T^*Q) \rightarrow \Gamma(L_h^0)(T^*Q)$  defined by  $\beta_+ = \beta^{ij} P_i \otimes dp_j$ . From a more general point of view, we may define Born reciprocity in para-Hermitian geometry as a morphism sending a  $B_-$ -transformation into a  $\beta_+$ -transformation, where a  $\beta_+$ -transformation is any map  $\beta_+ : \Gamma(L_v^0(T^*Q)) \rightarrow \Gamma(L_h^0(T^*Q))$  whose composition with the metric  $\eta_0$  is a 2-form.<sup>14</sup>  $\beta_+$ -transformations will be discussed more generally later on.

In terms of its action on the almost para-Hermitian structure  $(K_B, \eta_B)$  on  $T^*Q$ , Born reciprocity can be regarded as the change of polarization  $\vartheta = e^{\beta_+} e^{-B_-}$ :

$$\vartheta : K_B \xrightarrow{e^{-B_-}} K_0 \xrightarrow{e^{\beta_+}} K_\beta ,$$

where  $\beta$  is a bivector field of type  $(+0, -2)$ , or equivalently a 2-form of type  $(+2, -0)$  depending only on the fiber directions.<sup>15</sup> This change of polarization exchanges the role of the twisted distribution between the horizontal and vertical sub-bundle of the canonical para-Kähler structure: There is a new splitting  $T(T^*Q) = L_v^\beta(T^*Q) \oplus L_h^0(T^*Q)$ , where the vertical distribution  $L_v^0(T^*Q)$  is twisted to  $L_v^\beta(T^*Q)$  with

$$\Gamma(L_v^\beta(T^*Q)) = \text{Span}_{C^\infty(T^*Q)} \{ \tilde{D}^i = Q^i + \beta^{ij} P_j \} ,$$

while the horizontal distribution  $L_h^0(T^*Q)$  is unchanged. Thus the Lie algebra of the new splitting is

$$[P_i, P_j] = 0 , \quad [P_i, \tilde{D}^j] = 0 \quad \text{and} \quad [\tilde{D}^i, \tilde{D}^j] = (\tilde{\partial}^i \beta^{jk} - \tilde{\partial}^j \beta^{ik}) P_k .$$

The twisted sub-bundle is still maximally isotropic with respect to the metric  $\eta_0$ , thus  $(K_\beta, \eta_0)$  is an almost para-Hermitian structure on  $T^*Q$  (a  $\beta_+$ -transformation of the canonical para-Kähler structure as we showed above) with fundamental 2-form

$$\omega_\beta = dp_i \wedge dq^i + 2 \beta^{ij} dp_i \wedge dp_j = \omega_0 + 2 \tilde{b}_+ ,$$

where  $\tilde{b}_+ = \eta_0 \beta_+ = \beta^{ij} dp_i \wedge dp_j$  is a  $(+2, -0)$ -form with respect to the canonical splitting. The inverse of the 2-form  $\omega_\beta$  yields the local coordinate algebra with twisted Poisson brackets

$$\{q^i, q^j\}_\beta = 2 \beta^{ij}(p) , \quad \{q^i, p_j\}_\beta = \delta^i_j \quad \text{and} \quad \{p_i, p_j\}_\beta = 0 ,$$

which together with the non-vanishing Jacobiators

$$\{q^i, q^j, q^k\}_\beta = 3 R^{ijk}(p)$$

exhibit a nonassociative deformation of the configuration space  $Q$ .

This dynamical system is called the *R-flux model* and is purported to describe the phase space dynamics of closed strings propagating in locally non-geometric  $R$ -flux backgrounds [32, 39, 40, 42]. It is also possible to introduce a Born geometry on  $T^*Q$  in this setting, analogously to what we did

<sup>13</sup>For example, in  $d = 3$  dimensions the 2-form  $B_{ij} = \frac{1}{3} \rho \varepsilon_{ijk} q^k$ , which can be interpreted as a magnetic field sourced by a uniform distribution  $\rho$  of magnetic charges, is mapped to  $\beta^{ij} = \frac{1}{3} \rho \varepsilon^{ijk} p_k$ .

<sup>14</sup>In the present case a  $\beta_+$ -transformation is associated to a 2-form along the fibers, hence it is a  $(+2, -0)$ -form with respect to the canonical splitting.

<sup>15</sup>Here the bigrading is with respect to the canonical para-Kähler structure.

in Section 2.4: Starting from the generalized metric

$$\tilde{\mathcal{H}}_0 = \begin{pmatrix} g^{-1} & 0 \\ 0 & g \end{pmatrix}$$

with respect to the canonical polarization  $T(T^*\mathcal{Q}) = L_v^0(T^*\mathcal{Q}) \oplus L_h^0(T^*\mathcal{Q})$ , where  $g$  is a Riemannian metric on  $\mathcal{Q}$ , one computes the change of metric in the new  $\beta$ -twisted polarization to be

$$\tilde{\mathcal{H}}_{\beta+} = (e^{-\beta+})^\dagger \tilde{\mathcal{H}}_0 e^{-\beta+} = \begin{pmatrix} g^{-1} - \beta g \beta & -\beta g \\ g \beta & g \end{pmatrix}. \quad (4.4)$$

This is the correct global parameterization for the generalized metric in a non-geometric polarization familiar from generalized geometry and double field theory, which is a particular T-duality transformation of the generalized metric (2.37) of a geometric polarization with  $g_+ = g$  and  $b_+ = B$  [8,59,60].

Repeating the analysis of Section 4.1, we find that the  $\nabla^{\text{LC}}$ -bracket describes the emergence of  $R$ -flux as an obstruction to the weak integrability of  $L_v^\beta(T^*\mathcal{Q})$  with respect to the canonical para-Kähler structure:

$$[\tilde{D}^i, \tilde{D}^j]^{\nabla^{\text{LC}}} = R^{ijk} P_k,$$

while the horizontal distribution is weakly integrable. The D-bracket with respect to  $K_\beta$  vanishes, and the difference between the D-bracket and the  $\nabla^{\text{LC}}$ -bracket is measured by  $d\omega_\beta = 2 d\tilde{b}_+ = 2R$ .

Since the notion of flux in this context appears as an obstruction to relative integrability of two (almost) para-Hermitian structures, we can also compute the D-bracket associated to the almost para-complex structure  $K_B$  from (4.2) of the vectors spanning  $\Gamma(L_h^0(T^*\mathcal{Q}))$  and  $\Gamma(L_v^\beta(T^*\mathcal{Q}))$ , which will demonstrate how the fluxes thus far obtained can be generalized. The D-bracket associated with  $K_B$  is defined using the canonical para-Hermitian connection

$$\nabla^{\text{can}} = \Pi_+^B \nabla^{\text{LC}} \Pi_+^B + \Pi_-^B \nabla^{\text{LC}} \Pi_-^B$$

where

$$\begin{aligned} \Pi_+^B &= \frac{1}{2} (\mathbb{1} + K_B) = Q^i \otimes dp_i + B_{ij} Q^i \otimes dq^j, \\ \Pi_-^B &= \frac{1}{2} (\mathbb{1} - K_B) = P_i \otimes dq^i + B_{ij} Q^j \otimes dq^i, \end{aligned}$$

and  $\nabla^{\text{LC}}$  is the Levi-Civita connection of  $\eta_0$ . Then we obtain the D-brackets

$$[P_i, P_j]_B^{\text{D}} = \mathcal{H}_{ijk} \tilde{D}^k + \mathcal{F}_{ij}{}^k P_k \quad \text{and} \quad [\tilde{D}^i, \tilde{D}^j]_B^{\text{D}} = \mathcal{Q}^{ij}{}_k \tilde{D}^k + \mathcal{R}^{ijk} P_k,$$

where

$$\begin{aligned} \mathcal{H}_{ijk} &= -3 \partial_{[i} B_{jk]}, \\ \mathcal{F}_{ij}{}^k &= \beta^{km} \mathcal{H}_{mij}, \\ \mathcal{Q}^{ij}{}_k &= \beta^{im} \beta^{jl} \mathcal{H}_{mlk} + B_{km} \tilde{\partial}^m \beta^{ij}, \\ \mathcal{R}^{ijk} &= 3 \tilde{\partial}^{[i} \beta^{jk]} + 3 B_{lm} \beta^{[il} \tilde{\partial}^m \beta^{jk]} + \beta^{il} \beta^{jm} \beta^{kn} \mathcal{H}_{lmn}. \end{aligned}$$

These structure constants are precisely the generalized fluxes of double field theory associated to

an NS–NS background written in a holonomic basis (see e.g. [18]), after noticing that  $K_\beta$  can be obtained as a  $(-B_-)+\beta_+$ -deformation of  $K_B$ , and requiring that  $B$  and  $\beta$  do not depend on  $p_i$  and  $q^i$  respectively.

### 4.3 Fluxes from $B$ - and $\beta$ -Transformations

The  $R$ -flux model can be extended by considering more general  $B + \beta$ -transformations in order to formulate generalized fluxes as obstructions to compatibility between (almost) para-Hermitian structures. Generalized fluxes on a cotangent bundle may be interpreted in the sense of [18], where the cotangent bundle of an arbitrary manifold is the doubled target space of a membrane sigma-model for double field theory which involves geometric and non-geometric fluxes as components of a generalized Wess-Zumino term in the membrane action. Here we will see how the complete expressions of  $H$ -,  $f$ -,  $Q$ - and  $R$ -fluxes in double field theory emerge from suitable twists of the canonical para-Kähler structure on  $T^*\mathcal{Q}$ , in local coordinates  $(q^i, p_i)$ , for any  $d$ -dimensional manifold  $\mathcal{Q}$ .

**Geometric Fluxes from  $B$ -Transformations.** As a starting example, following [19] we may consider a slightly more general  $B$ -transformation than that considered in Section 4.1 by allowing a further dependence on the fiber coordinates  $p_i$ , and acting on the canonical para-Kähler structure on  $T^*\mathcal{Q}$  in the usual way. In this case the tensors  $(K_B, \eta_0, \omega_B)$  take the same forms as given in Section 4.1, with the differences that the Lie algebra of the two distributions is now

$$\begin{aligned} [D_i, D_j] &= (\partial_i B_{jk} - \partial_j B_{ik} + B_{il} \tilde{\partial}^l B_{jk} - B_{jl} \tilde{\partial}^l B_{ik}) Q^k, \\ [D_i, Q^j] &= -\tilde{\partial}^j B_{ik} Q^k, \\ [Q^i, Q^j] &= 0, \end{aligned} \tag{4.5}$$

and the flux of the fundamental 2-form becomes

$$d\omega_B = 2 (\partial_i B_{jk} + B_{im} \tilde{\partial}^m B_{jk}) dq^i \wedge dq^j \wedge dq^k + 2 \tilde{\partial}^i B_{jk} \tilde{\theta}_i \wedge dq^j \wedge dq^k, \tag{4.6}$$

when expressed in the splitting  $T(T^*\mathcal{Q}) = L_v^0(T^*\mathcal{Q}) \oplus L_h^B(T^*\mathcal{Q})$ . The closure of  $\omega_B$  is obstructed by a covariant  $H$ -flux and an  $f$ -flux with the bivector field  $\beta$  set to zero.

By (4.6) the bracket associated to the Levi-Civita connection also changes: With  $Z = Z^i D_i + \tilde{Z}_i Q^i \in \Gamma(T(T^*\mathcal{Q}))$ , we compute

$$\begin{aligned} \eta(\nabla_{D_i}^{\text{LC}} D_j - \nabla_{D_j}^{\text{LC}} D_i, Z) &= Z^k (\partial_i B_{jk} - \partial_j B_{ik} + B_{im} \tilde{\partial}^m B_{jk} - B_{jm} \tilde{\partial}^m B_{ik}), \\ \eta(\nabla_Z^{\text{LC}} D_i, D_j) &= Z^k (\partial_k B_{ij} + B_{km} \tilde{\partial}^m B_{ij}) + \tilde{Z}_k \tilde{\partial}^k B_{ij}, \end{aligned}$$

to obtain

$$\eta(\llbracket D_i, D_j \rrbracket^{\nabla^{\text{LC}}}, Z) = 3 Z^k (\partial_{[i} B_{jk]} + B_{[i|m} \tilde{\partial}^m B_{j]k}) + \tilde{Z}_k \tilde{\partial}^k B_{ij}. \tag{4.7}$$

This still agrees with (4.6) (up to the usual factor  $\frac{1}{2}$ ). The D-bracket again vanishes:

$$\eta(\llbracket D_i, D_j \rrbracket_B^{\text{D}}, Z) = 0,$$

so that the obstruction to (relative) weak integrability of  $L_h^B(T^*\mathcal{Q})$  is characterized by the components of the covariant  $H$ -flux, i.e. the  $H$ -flux without the section condition. The horizontal

component of the  $\nabla^{\text{LC}}$ -bracket in (4.7) takes the form of the  $f$ -flux when the bivector field  $\beta$  vanishes [18]. In other words, such fluxes also appear as the  $(+0, -3)$ -component of (4.6) with respect to  $K_0$ :

$$d\omega_B^{(+0, -3)_0} = 2 \left( \partial_i B_{jk} + B_{im} \tilde{\partial}^m B_{jk} \right) dq^i \wedge dq^j \wedge dq^k ,$$

and as the  $(+1, -2)$ -component

$$d\omega_B^{(+1, -2)_0} = 2 \tilde{\partial}^i B_{jk} dp_i \wedge dq^j \wedge dq^k .$$

In particular, this now implies that the Jacobiators  $\{q^i, p_j, p_k\}_B$  are also non-vanishing with the additional nonassociativity induced by the dependence of the  $f$ -flux on the dual fiber coordinates  $p_i$ . We can also see how the fluxes are related to the Lie algebra (4.5) of the maximally isotropic distributions  $L_{\mathbf{h}}^B(T^*\mathcal{Q})$  and  $L_{\mathbf{v}}^0(T^*\mathcal{Q})$ : The  $H$ -flux appears as the vertical component of the Lie bracket of two horizontal basis vectors, while the  $f$ -flux is exactly given by the vertical component of the bracket of a vertical and a horizontal basis vector. The vertical distribution remains weakly integrable with vanishing D-bracket and  $\nabla^{\text{LC}}$ -bracket; this is due to the vanishing of the non-geometric  $Q$ - and  $R$ -fluxes because  $\beta$  has been set to zero here. We have thus shown how geometric fluxes appear in the context of para-Hermitian geometry, and in particular as certain deformations of para-Hermitian structures; this provides an explicit example of the general global formulation of fluxes given in [19].

The lack of integrability in this case also means that  $b = \eta_0 B$  does not satisfy the Maurer-Cartan equation (2.31): The obstruction to integrability in (4.7) is given by

$$\left( d_+ b + (\wedge^3 \eta)[b, b]_-^{\text{S}} \right) (D_i, D_j, Z) = d\omega_B^{(+3, -0)_B} (D_i, D_j, Z) ,$$

where  $[b, b]_-^{\text{S}}$  is the Schouten-Nijenhuis bracket of  $b$  regarded as a bivector field and the bigrading on the right-hand side is with respect to the twisted para-complex structure  $K_B$ . The relation (2.32) between D-brackets is also easily verified in the present case, since the D-bracket associated to the canonical para-Kähler structure is the bracket associated to the Levi-Civita connection of the flat metric  $\eta_0$ .

**Non-Geometric Fluxes from  $\beta$ -Transformations.** The  $B$ -transformations considered above preserve the natural splitting induced by the projection map, i.e. they only twist the horizontal distribution while preserving the vertical distribution, since the cotangent projection does not uniquely define a horizontal sub-bundle. On the other hand, a skew transformation  $\beta$ , i.e.  $\eta_0 \beta$  is a 2-form, which does not preserve the natural splitting arising from the projection  $\pi : T^*\mathcal{Q} \rightarrow \mathcal{Q}$ , leads to the emergence of locally non-geometric  $R$ -flux, as we saw in the discussion of the  $R$ -flux model in Section 4.2. For this, we explain the notion of  $\beta$ -transformation on the cotangent bundle, in order to provide a more general interpretation of the  $R$ -flux model.

Let us consider the canonical para-Kähler structure  $(K_0, \eta_0)$  on the cotangent bundle  $T^*\mathcal{Q}$  for which  $T(T^*\mathcal{Q}) = L_{\mathbf{v}}^0(T^*\mathcal{Q}) \oplus L_{\mathbf{h}}^0(T^*\mathcal{Q})$ , and define a  $\beta$ -transformation on it by a map  $\beta : \Gamma(L_{\mathbf{v}}^0(T^*\mathcal{Q})) \rightarrow \Gamma(L_{\mathbf{h}}^0(T^*\mathcal{Q}))$ , so that in local coordinates  $\beta = \beta^{ij} P_j \otimes dp_i$ . The splitting is twisted to  $T(T^*\mathcal{Q}) = L_{\mathbf{v}}^{\beta}(T^*\mathcal{Q}) \oplus L_{\mathbf{h}}^0(T^*\mathcal{Q})$  such that  $\Gamma(L_{\mathbf{h}}^0(T^*\mathcal{Q})) = \text{Span}_{C^\infty(T^*\mathcal{Q})}\{P_i\}$  and  $\Gamma(L_{\mathbf{v}}^{\beta}(T^*\mathcal{Q})) = \text{Span}_{C^\infty(T^*\mathcal{Q})}\{\tilde{D}^i = Q^i + \beta^{ij} P_j\}$ , i.e. the almost para-complex structure

$$K_\beta = \tilde{D}^i \otimes dp_i - P_i \otimes \theta^i$$

is defined on  $T^*\mathcal{Q}$ , where  $dp_i$  and  $\theta^i = dq^i + \beta^{ij} dp_j$  are the dual 1-forms of the vectors  $\tilde{D}^i$  and  $P_i$  respectively. Then  $K_\beta$  is obtained as a  $\beta$ -transformation of  $K_0$ :

$$K_0 \longmapsto K_\beta = e^{-\beta} K_0 e^\beta ,$$

where

$$e^\beta = \begin{pmatrix} \mathbb{1} & \beta \\ 0 & \mathbb{1} \end{pmatrix} \in \mathcal{O}(d, d)(T^*\mathcal{Q}) ,$$

with respect to the canonical splitting  $T(T^*\mathcal{Q}) = L_v^0(T^*\mathcal{Q}) \oplus L_h^0(T^*\mathcal{Q})$ . In this case a  $\beta$ -transformation does not preserve verticality, hence it ruins the natural construction of a para-Hermitian structure on any bundle since the vertical sub-bundle is intrinsically defined by the underlying structures characterizing a bundle. This is analogous to what happens in generalized geometry, where there is a distinctive difference between  $B$ -transformations and  $\beta$ -transformations, since only  $B$ -transformations (with a closed 2-form) preserve the Courant bracket [1].

Since  $e^\beta \in \mathcal{O}(d, d)(T^*\mathcal{Q})$ , any  $\beta$ -transformation is an isometry of the flat metric  $\eta_0$ , hence  $L_v^\beta(T^*\mathcal{Q})$  and  $L_h^0(T^*\mathcal{Q})$  are still maximally isotropic distributions. This shows that the metric  $\eta_0$  can be written in local coordinates as

$$\eta_0 = \theta^i \otimes dp_i + dp_i \otimes \theta^i .$$

Thus  $\eta_0$  and  $K_\beta$  are compatible and together define an almost para-Hermitian structure. Then the fundamental 2-form reads

$$\omega_\beta = \eta_0 K_\beta = \omega_0 + 2 \eta_0 \beta = \theta^i \wedge dp_i ,$$

where  $2 \eta_0 \beta$  is a  $(+2, -0)$ -form with respect to the canonical splitting, and its exterior derivative is

$$d\omega_\beta = \partial_k \beta^{ij} \theta^k \wedge dp_j \wedge dp_i + (\tilde{\partial}^k \beta^{ij} + \beta^{km} \partial_m \beta^{ij}) dp_k \wedge dp_j \wedge dp_i . \quad (4.8)$$

From (4.8) we see that  $\omega_\beta$  fails to be closed and the obstruction to closure is given by  $\partial_k \beta^{ij}$ , which represents a globally non-geometric  $Q$ -flux when the  $B$ -field is turned off, and by  $\tilde{\partial}^k \beta^{ij} + \beta^{km} \partial_m \beta^{ij}$ , which is a locally non-geometric  $R$ -flux when  $B = 0$ .

We can now compute the D-bracket associated to  $K_\beta$  between two basis vectors  $\tilde{D}^i$ , since  $\llbracket P_i, P_j \rrbracket_\beta^D = 0$ , i.e.  $L_h^0(T^*\mathcal{Q})$  is a Frobenius integrable and weakly integrable eigenbundle. This computation relates the failure of closure of  $\omega_\beta$  to the relative concept of weak integrability of the twisted vertical distribution  $L_v^\beta(T^*\mathcal{Q})$  with respect to the canonical para-Kähler structure on  $T^*\mathcal{Q}$ . We first need the bracket associated to the (flat) Levi-Civita connection compatible with  $\eta_0$  which is given by

$$\eta_0(\llbracket \tilde{D}^i, \tilde{D}^j \rrbracket^{\nabla^{\text{LC}}}, Z) = Z^k \partial_k \beta^{ij} + 3 \tilde{Z}_k (\tilde{\partial}^{[k} \beta^{ij]} + \beta^{[k|m} \partial_m \beta^{ij]}) . \quad (4.9)$$

Thus from the bracket (4.9) and (4.8), we obtain  $\llbracket \tilde{D}^i, \tilde{D}^j \rrbracket_\beta^D = 0$ . In (4.9) we clearly see that the  $R$ -flux obstructs weak integrability while the  $Q$ -flux arises as the involutive component of the  $\nabla^{\text{LC}}$ -bracket, as expected on general grounds from the distinction between local versus global non-geometry.

**Generalized Fluxes from  $B+\beta$ -Transformations.** Finally we describe a twist of the canonical para-Kähler structure on  $T^*\mathcal{Q}$  given by the composition of a  $B$ -transformation and a  $\beta$ -transformation, as discussed in [18] in the context of generalized geometry. Following the discussion

above, the new almost para-Hermitian structure is given by

$$K = e^{-\beta} e^B K_0 e^{-B} e^\beta, \quad \eta = \eta_0 \quad \text{and} \quad \omega = \eta_0 K.$$

In matrix notation this reads

$$K = \begin{pmatrix} \mathbb{1} - 2\beta B & 2(\beta - \beta B \beta) \\ 2B & -\mathbb{1} + 2\beta B \beta \end{pmatrix}, \quad \eta_0 = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} \quad \text{and} \quad \omega = \begin{pmatrix} 2B & -\mathbb{1} + 2\beta B \beta \\ \mathbb{1} - 2\beta B & 2(\beta - \beta B \beta) \end{pmatrix}$$

where the fundamental 2-form can be written as

$$\omega = \omega_0 + 2\eta_0 (B\beta - \beta B) + 2\eta_0 B + 2\eta_0 (\beta - \beta B \beta).$$

The 2-form  $\omega_0 + 2\eta_0 (B\beta - \beta B)$  is its  $(+1, -1)$ -component,  $2\eta_0 B$  is its  $(+0, -2)$ -component, and  $2\eta_0 (\beta - \beta B \beta)$  is its  $(+2, -0)$ -component with respect to the canonical splitting  $T(T^*\mathcal{Q}) = L_v^0(T^*\mathcal{Q}) \oplus L_h^0(T^*\mathcal{Q})$ .

The splitting  $T(T^*\mathcal{Q}) = L_v^\beta(T^*\mathcal{Q}) \oplus L_h^B(T^*\mathcal{Q})$  is given in a local patch by

$$\begin{aligned} \Gamma(L_v^\beta(T^*\mathcal{Q})) &= \text{Span}_{C^\infty(T^*\mathcal{Q})} \{ \tilde{D}^i = Q^i + \beta^{ij} D_j \}, \\ \Gamma(L_h^B(T^*\mathcal{Q})) &= \text{Span}_{C^\infty(T^*\mathcal{Q})} \{ D_i = P_i + B_{ij} Q^j \}, \end{aligned}$$

with

$$\eta_0(D_i, D_j) = \eta_0(\tilde{D}^i, \tilde{D}^j) = 0 \quad \text{and} \quad \eta_0(D_i, \tilde{D}^j) = \eta_0(\tilde{D}^j, D_i) = \delta_i^j.$$

The Lie algebra defined by these distributions is

$$\begin{aligned} [D_i, D_j] &= J_{ijk} \tilde{D}^k - J_{ijm} \beta^{mk} D_k, \\ [D_i, \tilde{D}^j] &= C_i^{jk} D_k - W_{ik}^j \tilde{D}^k, \\ [\tilde{D}^i, \tilde{D}^j] &= P^{ij}_k \tilde{D}^k + A^{ijk} D_k, \end{aligned} \tag{4.10}$$

where

$$\begin{aligned} J_{ijk} &= 2(\partial_{[i} B_{j]k} + B_{[il} \tilde{\partial}^l B_{j]k}), \\ C_i^{jk} &= \partial_i \beta^{jk} + B_{il} \tilde{\partial}^l \beta^{jk} + \beta^{jm} \beta^{lk} (\partial_i B_{ml} + \partial_m B_{il} + B_{mp} \tilde{\partial}^p B_{il} - B_{ip} \tilde{\partial}^p B_{ml}), \\ W_{ik}^j &= \tilde{\partial}^j B_{ik} + B_{il} B_{mk} \tilde{\partial}^l \beta^{jm} + \beta^{jl} (\partial_i B_{lk} + \partial_l B_{ik} + B_{lm} \tilde{\partial}^m B_{ik} - B_{im} \tilde{\partial}^m B_{lk}), \\ P^{ij}_k &= 2(\beta^{p[i} \tilde{\partial}^{j]} B_{pk} - \beta^{l[i} \beta^{j]m} \partial_l B_{mk} - \beta^{l[i} \beta^{j]p} B_{lm} \tilde{\partial}^m B_{pk}), \\ A^{ijk} &= 2(\tilde{\partial}^{[i} \beta^{j]k} + \beta^{[il} \partial_l \beta^{j]k} + \beta^{[il} B_{lm} \tilde{\partial}^m \beta^{j]k}) - W^{ij}_l \beta^{lk}. \end{aligned} \tag{4.11}$$

Thus neither the vertical nor the horizontal distribution is integrable in this polarization. The dual 1-forms to  $\tilde{D}^i$  and  $D_i$  respectively are

$$\tilde{\theta}_i = dp_i + B_{ij} dq^j \quad \text{and} \quad \theta^i = dq^i + \beta^{ij} \tilde{\theta}_j.$$

Then we can write the local expressions of  $K$  and  $\eta_0$  as

$$K = \tilde{D}^i \otimes \tilde{\theta}_i - D_i \otimes \theta^i \quad \text{and} \quad \eta_0 = \tilde{\theta}_i \otimes \theta^i + \theta^i \otimes \tilde{\theta}_i,$$



so that the fundamental 2-form reads  $\omega = \eta_0 K = \theta^i \wedge \tilde{\theta}_i$ .

Generalized fluxes emerge from the bracket associated to the original canonical para-Kähler structure on  $T^*\mathcal{Q}$ . Hence we compute the  $\nabla^{\text{LC}}$ -bracket, which is the D-bracket of this structure, on the basis vectors of the twisted vertical and horizontal distributions:

$$[[D_i, D_j]]^{\nabla^{\text{LC}}} = \mathcal{H}_{ijk} \tilde{D}^k + \mathcal{F}_{ij}{}^k D_k \quad \text{and} \quad [[\tilde{D}^i, \tilde{D}^j]]^{\nabla^{\text{LC}}} = \mathcal{Q}^{ij}{}_k \tilde{D}^k + \mathcal{R}^{ijk} D_k ,$$

where

$$\begin{aligned} \mathcal{H}_{ijk} &= 3 \partial_{[i} B_{jk]} + 3 B_{[i|m} \tilde{\partial}^m B_{|jk]} , \\ \mathcal{F}_{ij}{}^k &= \tilde{\partial}^k B_{ij} + \beta^{km} \mathcal{H}_{mij} , \\ \mathcal{Q}^{ij}{}_k &= \partial_k \beta^{ij} + \beta^{im} \beta^{jl} \mathcal{H}_{mlk} + B_{km} \tilde{\partial}^m \beta^{ij} + 2 \beta^{p[i} \tilde{\partial}^{j]} B_{pk} , \\ \mathcal{R}^{ijk} &= 3 \tilde{\partial}^{[i} \beta^{jk]} + 3 \beta^{[i|l} \beta^{j|m} \tilde{\partial}^{k]} B_{ml} + 3 \beta^{[i|m} \partial_m \beta^{jk]} + 3 B_{lm} \beta^{[i|l} \tilde{\partial}^m \beta^{jk]} + \beta^{il} \beta^{jm} \beta^{kn} \mathcal{H}_{lmn} . \end{aligned}$$

These are precisely the generalized fluxes<sup>16</sup> of double field theory in a holonomic frame obtained from the standard Courant algebroid description [18, 61] or from the Roytenberg bracket [62]; their counterparts in an arbitrary non-holonomic frame can be obtained by further applying an  $\text{O}(d, d)(T^*\mathcal{Q})$ -transformation of the form (2.33) to write the change of basis  $\tilde{E}^a = (A^{-1})^a{}_i Q^i$  on  $L_{\mathbf{v}}^0$  and  $E_a = A_a{}^i P_i$  on  $L_{\mathbf{h}}^0$ , where  $A \in \text{End}(T\mathcal{Q})$  is a local  $\text{GL}(d, \mathbb{R})$ -transformation inducing geometric  $f$ -flux through the non-vanishing Lie brackets  $[E_i, E_j] = f_{ij}{}^k E_k$  with  $f_{ij}{}^k = (A^{-1})^a{}_{[i} (A^{-1})^b{}_{j]} \partial_a A_b{}^k$ . In the present case these fluxes arise as a measure of the relative weak integrability between two (almost) para-Hermitian structures related via a composition of a  $B$ -transformation and a  $\beta$ -transformation,<sup>17</sup> with the  $H$ -flux obstructing the integrability of  $L_{\mathbf{h}}^B$  and the  $R$ -flux obstructing integrability of  $L_{\mathbf{v}}^{\beta}$ . This also justifies once more the choice in [18] of the cotangent bundle for the doubled target space of the membrane sigma-model for double field theory: It carries a natural para-Kähler structure obtained solely from the properties of the bundle itself, while the generalized fluxes can be encoded in its deformation and used to introduce a Wess-Zumino type topological term in the membrane action. Such a construction can of course also be carried out on any flat para-Kähler manifold, with the same result; we have chosen the cotangent bundle because it naturally carries such a structure.

The D-bracket associated to the  $B + \beta$ -twisted para-Hermitian structure gives only the integrable part of the  $\nabla^{\text{LC}}$ -bracket algebra defined by the two distributions:

$$[[D_i, D_j]]^{\text{D}} = J_{ijm} \beta^{mk} D_k \quad \text{and} \quad [[\tilde{D}^i, \tilde{D}^j]]^{\text{D}} = P^{ij}{}_k \tilde{D}^k .$$

These brackets also show that the fundamental 2-form  $\omega$  is not closed; the components of  $d\omega$  can be obtained directly from  $\omega = \theta^i \wedge \tilde{\theta}_i$ , or equivalently as the difference of the D-bracket and the  $\nabla^{\text{LC}}$ -bracket by using (2.25) to get

$$\begin{aligned} d\omega &= 2 \left( \mathcal{H}_{ijk} \theta^i \wedge \theta^j \wedge \theta^k - \mathcal{R}^{ijk} \tilde{\theta}_i \wedge \tilde{\theta}_j \wedge \tilde{\theta}_k \right. \\ &\quad \left. + (\mathcal{F}_{ij}{}^k - J_{ijm} \beta^{mk}) \theta^i \wedge \theta^j \wedge \tilde{\theta}_k - (\mathcal{Q}^{ij}{}_k - P^{ij}{}_k) \tilde{\theta}_i \wedge \tilde{\theta}_j \wedge \theta^k \right) . \end{aligned}$$

<sup>16</sup>The Bianchi identities for the fluxes follow from the Jacobi identity for the Lie brackets (4.10).

<sup>17</sup>Note that an arbitrary change of polarization  $\vartheta \in \text{O}(d, d)(M)$  can be parameterized as  $\vartheta = e^{-B} A e^{\beta}$ , cf. Interlude 2.11.

The generalized fluxes then appear amongst the Jacobiators of the corresponding twisted Poisson brackets

$$\{q^i, q^j\}_{B,\beta} = 2(\beta^{ij} - \beta^{ik} B_{kl} \beta^{lj}) , \quad \{q^i, p_j\}_{B,\beta} = \delta^i_j - 2\beta^{ik} B_{kj} \quad \text{and} \quad \{p_i, p_j\}_{B,\beta} = 2B_{ij} .$$

Finally, we can start from a reference generalized metric (4.4) and apply the change of polarization

$$\begin{aligned} \tilde{\mathcal{H}}_{B,\beta} &= (e^{-\beta})^\dagger (e^{-B})^\dagger \tilde{\mathcal{H}}_0 e^{-B} e^{-\beta} \\ &= \begin{pmatrix} g^{-1} - \beta g \beta + (g^{-1} B \beta)_{\text{ss}} + \beta B g^{-1} G \beta & g^{-1} B - \beta g + \beta B g^{-1} B \\ -B g^{-1} + g \beta - B g^{-1} B \beta & g - B g^{-1} B \end{pmatrix} , \end{aligned}$$

where the subscript  $\text{ss}$  means the skew-symmetric part of a  $(0,2)$ -tensor on  $M = T^*\mathcal{Q}$ ; this is indeed the correct form of the covariant generalized metric on  $T^*\mathcal{Q}$  [18].

## 5 Para-Hermitian Geometry of Drinfel'd Doubles

We shall now move on to study other related examples of how fluxes arise in para-Hermitian geometry, which extend our previous considerations globally to certain classes of parallelizable manifolds. A broad class of examples of para-Hermitian manifolds naturally arises in the form of Lie groups that are Drinfel'd doubles, which provide a global extension of the local geometry of the cotangent bundles we considered previously to curved backgrounds; from a dynamical perspective, they describe duality transformations between particular field theories valued in Lie groups which are given by principal chiral models. At the same time, they automatically capture the relation with generalized geometry and provide a natural notion of non-abelian T-duality for Lie groups. Double field theory and in particular Poisson-Lie T-duality on Drinfel'd doubles has also been considered by [34, 35, 63, 64]. In this section we will adapt the description presented in [35] to the formalism of Sections 2 and 4 in this setting.

### 5.1 The Left-Invariant Para-Hermitian Structure

A (classical) Drinfel'd double  $\mathcal{D}$  is a  $2d$ -dimensional Lie group whose Lie algebra  $\mathfrak{d}$  can be given in the split form  $\mathfrak{d} = \mathfrak{g} \ltimes \tilde{\mathfrak{g}}$ , where  $\mathfrak{g}$  and  $\tilde{\mathfrak{g}}$  are two dual Lie subalgebras of  $\mathfrak{d}$  generated, respectively, by elements  $T_i$  and  $\tilde{T}^i$  with  $i = 1, \dots, d$  satisfying

$$[T_i, T_j] = f_{ij}{}^k T_k , \quad [T_i, \tilde{T}^j] = f_{ki}{}^j \tilde{T}^k - Q^{kj}{}_i T_k \quad \text{and} \quad [\tilde{T}^i, \tilde{T}^j] = Q^{ij}{}_k \tilde{T}^k . \quad (5.1)$$

The Jacobi identity for the Lie bracket implies the algebraic Bianchi identities

$$f_{[ij}{}^m f_{k]m}{}^n = 0 , \quad f_{ij}{}^m Q^{kl}{}_m = Q^{m[k}{}_i f_{j]m}{}^l \quad \text{and} \quad Q^{[ij}{}_m Q^{k]m}{}_n = 0 .$$

The Lie subalgebras  $\mathfrak{g}$  and  $\tilde{\mathfrak{g}}$  together define a Lie bialgebra  $(\mathfrak{g}, \tilde{\mathfrak{g}})$ , and they respectively generate two Lie subgroups  $\mathcal{G}$  and  $\tilde{\mathcal{G}}$  of  $\mathcal{D}$  such that  $\mathcal{D} = \mathcal{G} \ltimes \tilde{\mathcal{G}}$  which are dual in the sense that their Lie algebras are dual. This duality induces a natural  $\text{Ad}(\mathcal{D})$ -invariant inner product on the Lie algebra of the Drinfel'd double. The Lie brackets (5.1) describe the gauge algebra of a string compactification on a Poisson-Lie background (see e.g. [63]).

This is not the only possible splitting of the Lie algebra  $\mathfrak{d}$ . Generally, a splitting of  $\mathfrak{d}$  given in

terms of two maximally isotropic subspaces is called a *polarization*.<sup>18</sup> When the polarization is given in terms of two maximally isotropic Lie subalgebras  $\mathfrak{g}$  and  $\tilde{\mathfrak{g}}$  as above, the triple  $(\mathfrak{d}, \mathfrak{g}, \tilde{\mathfrak{g}})$  is called a *Manin triple*. In this case the duality pairing between  $\mathfrak{g}$  and  $\tilde{\mathfrak{g}}$  can be regarded as an invariant  $O(d, d)$ -metric  $\eta$  on the Lie algebra  $\mathfrak{d}$  such that

$$\eta(T_i, T_j) = \eta(\tilde{T}^i, \tilde{T}^j) = 0 \quad \text{and} \quad \eta(T_i, \tilde{T}^j) = \eta(\tilde{T}^j, T_i) = \delta_i^j .$$

The splitting of  $\mathfrak{d}$  can be equivalently regarded as an invariant para-complex structure  $K$  on  $\mathfrak{d}$  such that  $\mathfrak{g}$  is its  $+1$ -eigenspace and  $\tilde{\mathfrak{g}}$  is its  $-1$ -eigenspace, so that

$$K = T_i \otimes \tilde{T}^i - \tilde{T}^i \otimes T_i .$$

This is the para-Kähler structure associated to the Manin triple polarization  $\mathfrak{d} = \mathfrak{g} \ltimes \tilde{\mathfrak{g}}$ ; alternatively, the para-Kähler structure  $-K$  is associated to the polarization  $\mathfrak{d} = \tilde{\mathfrak{g}} \ltimes \mathfrak{g}$ . In the spirit of Section 4.2, the change of polarization  $K \mapsto -K$  is a type of Born reciprocity transformation.

The Lie algebra  $\mathfrak{d}$  is isomorphic to the Lie algebra of left-invariant vector fields on the Drinfel'd double  $D$ , which are globally defined because the tangent and cotangent bundles of a Lie group are trivial vector bundles. Hence we may translate this construction to the group manifold  $D$ , where coordinates are given by  $x^I = (x^i, \tilde{x}_i)$  in a local chart. For this, we need to construct left-invariant 1-forms and vector fields. We first describe this construction in a Manin triple polarization, and then later on give the general form in an arbitrary polarization.

In order to obtain the local expression of the left-invariant 1-forms, we fix the Iwasawa decomposition of  $D$  to be  $\gamma = g \tilde{g}$ , for any element  $\gamma \in D$  in terms of elements  $g = \exp(x^i T_i) \in G$  and  $\tilde{g} = \exp(\tilde{x}_i \tilde{T}^i) \in \tilde{G}$ ; an equivalent discussion is possible with the dual Iwasawa decomposition  $\gamma = \tilde{g} g$ . Then any left-invariant 1-form on  $D$  is valued in  $\mathfrak{d} \otimes T^*D$  and has the expression

$$\Theta = \gamma^{-1} d\gamma = \tilde{g}^{-1} g^{-1} d(g \tilde{g}) = \tilde{g}^{-1} (g^{-1} dg) \tilde{g} + \tilde{g}^{-1} (d\tilde{g} \tilde{g}^{-1}) \tilde{g} ,$$

where  $\lambda = g^{-1} dg = \lambda^m T_m$  (depending only on the coordinates  $x^i$ ) and  $\tilde{\rho} = d\tilde{g} \tilde{g}^{-1} = \tilde{\rho}_m \tilde{T}^m$  (depending only on the coordinates  $\tilde{x}_i$ ) are Lie algebra-valued left- and right-invariant 1-forms on  $G$  and  $\tilde{G}$  respectively. Then the left-invariant 1-form can be written as

$$\Theta = \lambda^m (\tilde{g}^{-1} T_m \tilde{g}) + \tilde{\rho}_m (\tilde{g}^{-1} \tilde{T}^m \tilde{g}) .$$

This shows that we need the adjoint action of  $\tilde{G}$  on the generators  $T_M = \left(\begin{smallmatrix} T_m \\ \tilde{T}^m \end{smallmatrix}\right)$  of  $\mathfrak{d}$ , with the Lie brackets  $[T_M, T_N] = t_{MN}{}^P T_P$ , which has the form [35]

$$\tilde{g}^{-1} \begin{pmatrix} T_m \\ \tilde{T}^m \end{pmatrix} \tilde{g} = \begin{pmatrix} (\tilde{A}^{-1})_m{}^n & \tilde{b}_{mn} \\ 0 & \tilde{A}^m{}_n \end{pmatrix} \begin{pmatrix} T_n \\ \tilde{T}^n \end{pmatrix} .$$

Here the block matrices are defined by the adjoint action and they all depend only on the coordinates  $\tilde{x}_i$ , where  $\tilde{b}_{mn}$  is skew-symmetric because we have chosen a Manin triple polarization. Therefore the left-invariant 1-form is

$$\Theta = \lambda^m (\tilde{A}^{-1})_m{}^n T_n + (\lambda^m \tilde{b}_{mn} + \tilde{\lambda}_n) \tilde{T}^n ,$$

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<sup>18</sup>See [65, 66] for a more precise definition and comprehensive treatment of the geometry of Drinfel'd doubles.

where in the second term on the right-hand side we used  $\tilde{g}^{-1} \tilde{\rho} \tilde{g} = \tilde{\lambda}$ , where  $\tilde{\lambda} = \tilde{g}^{-1} d\tilde{g} = \tilde{\lambda}_m \tilde{T}^m$ , hence  $\tilde{\rho}_m \tilde{A}^m_n = \tilde{\lambda}_n$ . The Lie algebra components of  $\Theta = \Theta^M T_M$  are given by the 1-forms

$$\Theta^M = \begin{pmatrix} \Theta^m \\ \tilde{\Theta}_m \end{pmatrix} = \begin{pmatrix} \lambda_i^n (\tilde{A}^{-1})_n^m dx^i \\ \lambda_i^n \tilde{b}_{nm} dx^i + \tilde{\lambda}_m^i d\tilde{x}_i \end{pmatrix} \quad (5.2)$$

from which we obtain the dual left-invariant vector fields

$$Z_M = \begin{pmatrix} Z_m \\ \tilde{Z}^m \end{pmatrix} = \begin{pmatrix} \tilde{A}^n_m ((\lambda^{-1})_n^i \frac{\partial}{\partial x^i} - (\tilde{\lambda}^{-1})_i^p \tilde{b}_{np} \frac{\partial}{\partial \tilde{x}_i}) \\ (\tilde{\lambda}^{-1})_i^m \frac{\partial}{\partial \tilde{x}_i} \end{pmatrix}. \quad (5.3)$$

The Lie brackets  $[Z_M, Z_N] = t_{MN}^P Z_P$  close to the Lie algebra  $\mathfrak{d}$  from (5.1).

We can now construct the left-invariant para-Hermitian structure induced by the para-Hermitian structure defined on the Lie algebra  $\mathfrak{d}$  in a Manin triple polarization; see also [34] for a similar discussion of this construction. The (globally defined) left-invariant para-complex structure is given by

$$K = Z_m \otimes \Theta^m - \tilde{Z}^m \otimes \tilde{\Theta}_m = \frac{\partial}{\partial x^i} \otimes dx^i - 2 \tilde{b}_{pn} (\tilde{\lambda}^{-1})_i^n \lambda_j^p \frac{\partial}{\partial \tilde{x}_i} \otimes dx^j - \frac{\partial}{\partial \tilde{x}_i} \otimes d\tilde{x}_i,$$

where the eigenbundles  $L_+$  and  $L_-$  of  $K$  are both integrable since they are generated, respectively, by the vector fields  $Z_m$  and  $\tilde{Z}^m$  which close to the Lie subalgebras  $\mathfrak{g}$  and  $\tilde{\mathfrak{g}}$  from (5.1).

The left-invariant metric with Lorentzian signature induced by the duality pairing is

$$\eta = \Theta^m \otimes \tilde{\Theta}_m + \tilde{\Theta}_m \otimes \Theta^m.$$

In this splitting (polarization), it can be regarded as the  $O(d, d)$ -invariant constant metric

$$\eta = \begin{pmatrix} 0 & \mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix}.$$

In the local coordinates  $x^I = (x^i, \tilde{x}_i)$  on the group manifold  $D$  it is given by

$$\eta = \lambda_i^m (\tilde{A}^{-1})_m^n \tilde{\lambda}_n^j (dx^i \otimes d\tilde{x}_j + d\tilde{x}_j \otimes dx^i), \quad (5.4)$$

since  $\tilde{b}_{mn}$  is skew-symmetric.

Finally, the left-invariant fundamental 2-form is thus given by

$$\omega = \eta K = \tilde{\Theta}_m \wedge \Theta^m.$$

It can be shown that  $\omega$  is not closed:

$$d\omega = d\tilde{\Theta}_m \wedge \Theta^m - \tilde{\Theta}_m \wedge d\Theta^m.$$

Using the Maurer-Cartan structure equations

$$d\Theta^P = -\frac{1}{2} t_{MN}^P \Theta^M \wedge \Theta^N,$$

which in polarized components read

$$\begin{aligned} d\Theta^p &= -\frac{1}{2} f_{mn}{}^p \Theta^m \wedge \Theta^n - Q^{mp}{}_n \Theta^n \wedge \tilde{\Theta}_m , \\ d\tilde{\Theta}_p &= -f_{mp}{}^n \tilde{\Theta}_n \wedge \Theta^m - \frac{1}{2} Q^{mn}{}_p \tilde{\Theta}_m \wedge \tilde{\Theta}_n , \end{aligned}$$

we obtain

$$d\omega = -\frac{1}{2} (f_{mn}{}^p \tilde{\Theta}_p \wedge \Theta^m \wedge \Theta^n - Q^{mn}{}_p \Theta^p \wedge \tilde{\Theta}_m \wedge \tilde{\Theta}_n) .$$

We have therefore shown that a Drinfel'd double  $D$  is endowed with a natural para-Hermitian structure, having two Lagrangian foliations with leaves given by  $G$  and  $\tilde{G}$ ; this result does not depend on the choice of Iwasawa decomposition of  $D$ . It is para-Kähler if and only if both groups  $G$  and  $\tilde{G}$  (and hence  $D$ ) are abelian. Since  $\tilde{\mathfrak{g}}$  is dual to  $\mathfrak{g}$ , there is a bundle isomorphism  $T^*G \simeq T\tilde{G}$ , and the splitting  $TD \simeq TG \oplus T\tilde{G}$  appropriate to double field theory is naturally identified with the splitting  $TD \simeq \mathbb{T}G$  appropriate to generalized geometry on the Lie group  $G$ . If  $G$  is connected and simply connected, then the Lie bialgebra  $(\mathfrak{g}, \tilde{\mathfrak{g}})$  makes  $G$  into a Poisson-Lie group [67] and endows the pair  $(TG, T^*G)$  with the structure of a Lie bialgebroid, while the Drinfel'd double structure  $\mathfrak{d} = \mathfrak{g} \ltimes \tilde{\mathfrak{g}}$  makes the generalized tangent bundle  $\mathbb{T}G = TG \oplus T^*G$  into a Courant algebroid. Generally, a Manin triple polarization of  $\mathfrak{d}$  gives a decomposition of  $TD$  into Dirac structures, i.e. integrable maximally isotropic sub-bundles of  $TD$ .

Thus for a Drinfel'd double, doubled geometry naturally coincides with generalized geometry. In this case we recover  $\mathcal{F}_+ = G$  from the Manin triple  $(\mathfrak{d}, \mathfrak{g}, \tilde{\mathfrak{g}})$  as the physical spacetime, identified as the coset  $G = D/\tilde{G}$  by the left action of the subgroup  $\tilde{G}$  whose isotropy group  $\tilde{G}$  is generated by the right action. Alternatively, using instead the Manin triple  $(\mathfrak{d}, \tilde{\mathfrak{g}}, \mathfrak{g})$  recovers  $\mathcal{F}_+ = \tilde{G} = D/G$  as the physical spacetime, and the process of exchanging the Manin triple polarizations  $(\mathfrak{d}, \mathfrak{g}, \tilde{\mathfrak{g}})$  and  $(\mathfrak{d}, \tilde{\mathfrak{g}}, \mathfrak{g})$  is often called *Poisson-Lie T-duality* [63, 68]. A change in a Manin triple polarization of  $\mathfrak{d}$  is also called a *non-abelian T-duality* [34].

## 5.2 Manin Triples as Flux Deformations of Para-Kähler Structures

We shall now give a different geometric interpretation of this construction in terms of deformations of a fixed reference para-Kähler structure on the group manifold  $D$ , similarly to what we did in Section 4. From the local coordinate expression (5.4) of the left-invariant metric  $\eta$ , we see that it can be obtained as a transformation of the  $O(d, d)(D)$ -metric  $\eta_0 = dx^i \otimes d\tilde{x}_i + d\tilde{x}_i \otimes dx^i$  on the group manifold via a sequence of transformations; however, in contrast to the situation of Section 4, these transformations are generally *not* valued in the T-duality group  $O(d, d)(D)$ . From this perspective we may infer that, since the group manifold  $D$  is  $2d$ -dimensional, a flat para-Kähler structure can always be defined by

$$K_0 = \frac{\partial}{\partial x^i} \otimes dx^i - \frac{\partial}{\partial \tilde{x}_i} \otimes d\tilde{x}_i \quad \text{and} \quad \eta_0 = dx^i \otimes d\tilde{x}_i + d\tilde{x}_i \otimes dx^i ,$$

with abelian eigenbundles of  $K_0$  spanned by the vector fields  $\frac{\partial}{\partial x^i}$  and  $\frac{\partial}{\partial \tilde{x}_i}$ ; the canonical para-Hermitian connection of this structure is the Levi-Civita connection of  $\eta_0$ , and the fundamental 2-form is the closed form  $\omega_0 = \eta_0 K_0 = dx^i \wedge d\tilde{x}_i$ .

Then the para-Hermitian structure  $(K, \eta)$  induced by a Manin triple polarization of the Lie

algebra  $\mathfrak{d}$  is a deformation of  $(K_0, \eta_0)$  given by a composition of three types of transformations:

- $\mathbf{GL}(d, \mathbb{R})_+ \times \mathbf{GL}(d, \mathbb{R})_-$ -transformations. Such a transformation of the trivial para-Kähler structure  $(K_0, \eta_0)$  is given by

$$K' = K_0 = \frac{\partial}{\partial x^i} \otimes dx^i - \frac{\partial}{\partial \tilde{x}_i} \otimes d\tilde{x}_i \quad \text{and} \quad \eta' = \lambda_i^m \tilde{\lambda}_m^j (dx^i \otimes d\tilde{x}_j + d\tilde{x}_j \otimes dx^i) .$$

The corresponding vector fields spanning the two maximally isotropic distributions are

$$Z'_m = (\lambda^{-1}(x))_m^i \frac{\partial}{\partial x^i} \quad \text{and} \quad \tilde{Z}'^m = (\tilde{\lambda}^{-1}(\tilde{x}))_i^m \frac{\partial}{\partial \tilde{x}_i} ,$$

whose dual 1-forms are obtained, respectively, from

$$\lambda = g^{-1} dg = \lambda_i^m dx^i T_m = \Theta'^m T_m \quad \text{and} \quad \tilde{\lambda} = \tilde{g}^{-1} d\tilde{g} = \tilde{\lambda}_m^i d\tilde{x}_i \tilde{T}^m = \tilde{\Theta}'_m \tilde{T}^m .$$

It follows that the vectors  $Z'_m$  and  $\tilde{Z}'^m$  spanning the two distributions close, respectively, to the Lie algebras  $\mathfrak{g}$  and  $\tilde{\mathfrak{g}}$  such that  $[Z'_m, \tilde{Z}'^n] = 0$ , which means that they separately give two foliations of  $D$  with the subgroups  $G$  and  $\tilde{G}$  as leaves. From this point of view, despite the fact that  $Z'_m$  and  $\tilde{Z}'^m$  close to the Lie algebras defining the Drinfel'd double, they are not obtained as global left-invariant vector fields on  $D$ . The fundamental 2-form transforms to  $\omega' = \lambda^m \wedge \tilde{\lambda}_m$ , which is not closed:

$$d\omega' = -\frac{1}{2} (f_{pq}^m \tilde{\lambda}_m \wedge \lambda^p \wedge \lambda^q - Q^{pq}_m \lambda^m \wedge \tilde{\lambda}_p \wedge \tilde{\lambda}_q) ,$$

where we used the Maurer-Cartan structure equations for the Lie groups  $G$  and  $\tilde{G}$ . Thus the non-closure of the fundamental 2-form is, in this framework, related to the non-abelian nature of the Lie algebras of the distributions. We stress that this choice is fundamental, since it gives two canonical “dual” foliations, one of which can be interpreted as the physical spacetime submanifold. In this way the duality plays a fundamental role in linking the description of a physical spacetime to its dual.

- $B$ -transformations. Denoted here  $\tilde{b}$ , these transformations act to give

$$\begin{pmatrix} Z''_m \\ \tilde{Z}''^m \end{pmatrix} = \begin{pmatrix} \delta_m^n & \tilde{b}_{nm} \\ 0 & \delta^m_n \end{pmatrix} \begin{pmatrix} Z'_n \\ \tilde{Z}'^n \end{pmatrix} ,$$

which yields the globally defined left-invariant vector fields

$$Z_M = \begin{pmatrix} Z_m \\ \tilde{Z}^m \end{pmatrix} = \begin{pmatrix} ((\lambda^{-1})_m^i \frac{\partial}{\partial x^i} - (\tilde{\lambda}^{-1})_i^p \tilde{b}_{mp} \frac{\partial}{\partial \tilde{x}_i}) \\ (\tilde{\lambda}^{-1})_i^m \frac{\partial}{\partial \tilde{x}_i} \end{pmatrix} .$$

Thus from this perspective a  $B$ -transformation acts by twisting the two distributions, with  $K = e^{-B} K_0 e^B$ , and preserving the metric  $\eta'$ , i.e. it is an  $O(d, d)(D)$ -transformation. The fundamental 2-form  $\omega''$  is not closed and  $d\omega''$  has the same coefficients as  $d\omega'$ , written in the new dual basis.

- $\mathbf{GL}(d, \mathbb{R})_+$ -transformations. Such a transformation rotates the distribution spanned by  $Z''_m$  while preserving the other:

$$\begin{pmatrix} Z_m \\ \tilde{Z}^m \end{pmatrix} = \begin{pmatrix} \tilde{A}_m^n & 0 \\ 0 & \delta^m_n \end{pmatrix} \begin{pmatrix} Z''_n \\ \tilde{Z}''^n \end{pmatrix} ,$$

giving finally the basis (5.3). The combined action of the last two transformations allows for non-

vanishing brackets  $[Z'_m, \tilde{Z}'^n]$  and does not affect the Lie algebras closed by the vector fields  $Z'_m$  and  $\tilde{Z}'^m$ , thus giving the desired geometric interpretation of how the Manin triple polarization is obtained in terms of deformations of para-Hermitian structures. This last transformation does not change the para-complex structure  $K$ , which is affected only by a  $B$ -transformation, while the Lorentzian metric is finally transformed into the local expression (5.4). Again the fundamental 2-form  $\omega$  is not closed, as shown previously, and  $d\omega$  still has the same components as  $d\omega'$ . Thus the last two transformations preserve the 3-form  $d\omega'$  in a Manin triple polarization.

We have thus shown that a  $\mathrm{GL}(d, \mathbb{R})_+ \times \mathrm{GL}(d, \mathbb{R})_-$ -transformation fixes the Lie algebra closed by the two distributions separately, thereby governing the closure of the fundamental 2-form. A  $B$ -transformation (or  $\beta$ -transformation) is needed to twist the two distributions together, as seen from its action on the para-complex structure which deforms  $K_0$  into  $K$ . This means that any time there is a  $B$ - or  $\beta$ -transformation involved in this deformation, we will find a non-trivial Kalb-Ramond field on the physical spacetime submanifold; this is not the same as having non-vanishing generalized fluxes, since fluxes are governed by the non-integrability of the chosen polarization as we saw in Section 4. Finally, a  $\mathrm{GL}(d, \mathbb{R})_+$ -transformation does not affect the para-complex structure  $K$  but is needed to obtain the complete Lie algebra  $\mathfrak{d}$ . Note that, using the dual Iwasawa decomposition, the second transformation becomes a  $\beta$ -transformation while the last transformation becomes a  $\mathrm{GL}(d, \mathbb{R})_-$ -transformation rotating only the distribution spanned by  $\tilde{Z}''^m$ , and giving the same final result. Later on we will give an explicit example of this latter approach in the Drinfel'd double description of the cotangent bundle  $T^*\mathbf{G}$  of a Lie group  $\mathbf{G}$ .

Let us now show how fluxes arise from the para-Hermitian geometry of Drinfel'd doubles. In order to compute the D-bracket associated to the Lie algebra induced para-Hermitian structure  $(K, \eta)$  and the  $\nabla^{\mathrm{LC}}$ -bracket compatible with the undeformed trivial para-Kähler structure  $(K_0, \eta_0)$  on  $\mathbf{D}$ , we need the connection coefficients in the non-holonomic frame  $Z_M$ , which we compute to be

$$\begin{aligned} \Gamma^i_{jk} &= \frac{1}{2} f_{kj}^i, & \Gamma^{ij}_k &= \frac{1}{2} Q^{ji}_k, & \Gamma_{ij}^k &= \frac{1}{2} f_{ij}^k & \text{and} & & \Gamma^{ijk} &= 0, \\ \Gamma_i{}^j{}_k &= \frac{1}{2} f_{ik}^j, & \Gamma_i{}^{jk} &= \frac{1}{2} Q^{kj}_i, & \Gamma^i{}_j{}^k &= \frac{1}{2} Q^{ik}_j & \text{and} & & \Gamma_{ijk} &= 0. \end{aligned}$$

Thus the  $\nabla^{\mathrm{LC}}$ -bracket is

$$[[Z_m, Z_n]]^{\nabla^{\mathrm{LC}}} = \frac{1}{2} f_{mn}^k Z_k \quad \text{and} \quad [[\tilde{Z}^m, \tilde{Z}^n]]^{\nabla^{\mathrm{LC}}} = \frac{1}{2} Q^{mn}_k \tilde{Z}^k$$

while the D-bracket is

$$[[Z_m, Z_n]]^{\mathbf{D}} = f_{mn}^k Z_k \quad \text{and} \quad [[\tilde{Z}^m, \tilde{Z}^n]]^{\mathbf{D}} = Q^{mn}_k \tilde{Z}^k.$$

From these brackets it is clear that the left-invariant para-Hermitian structure  $(K, \eta)$  is compatible with the trivial para-Kähler structure  $(K_0, \eta_0)$  on  $\mathbf{D}$ , since  $(K, \eta)$  is weakly integrable with respect to the D-bracket of  $(K_0, \eta_0)$ . This holds in any Manin triple polarization of  $\mathbf{D}$ .

The difference between these brackets is measured by  $d\omega$ . The lack of closure of the fundamental 2-form  $\omega$  can be interpreted as a transformation from two abelian integrable distributions (the eigenbundles of the trivial para-Kähler structure on  $\mathbf{D}$ ) to two non-abelian integrable distributions (the eigenbundles of the left-invariant para-Hermitian structure on  $\mathbf{D}$ ). As  $d\omega$  is characterized by the structure constants of the Lie algebras closed by the two distributions, this impinges on the

difference between the D-bracket and the  $\nabla^{\text{LC}}$ -bracket. This can be regarded as a situation in which only (globally geometric)  $f$ -flux and (locally geometric)  $Q$ -flux arise. The  $\nabla^{\text{LC}}$ -bracket does not give any  $H$ -flux or  $R$ -flux in a Manin triple polarization, hence in order to induce such fluxes it is necessary to choose a polarization with non-integrable distributions, at least if the reference structure is the trivial para-Kähler structure; we consider this construction in detail below. Alternatively, one can accordingly change the reference para-Hermitian structure.

No matter what the polarization, in order to recover the physical background fields on the spacetime submanifold it is necessary to introduce a Born geometry  $(K, \eta, \mathcal{H})$  that is left-invariant and globally defined on  $D$ . Following [35], this can be achieved by the introduction of the globally defined (left-invariant and Riemannian) generalized metric

$$\mathcal{H} = \delta_{MN} \Theta^M \otimes \Theta^N = \delta_{mn} \Theta^m \otimes \Theta^n + \delta^{mn} \tilde{\Theta}_m \otimes \tilde{\Theta}_n$$

on the Drinfel'd double  $D$ .

**Example 5.5.** A non-trivial example of a Born geometry for Drinfel'd doubles is given by  $D = \text{SL}(2, \mathbb{C})$ , regarded as a six-dimensional real Lie group; this is studied in [69] in the context of the isotropic rigid rotator, whose configuration space is the Lie group  $\text{SU}(2)$ , as an alternative carrier manifold (to the tangent bundle) for the lift of the dynamics. It has a Manin triple polarization  $\text{SL}(2, \mathbb{C}) = \text{SU}(2) \ltimes \text{SB}(2, \mathbb{C})$ .<sup>19</sup> In a suitable basis of  $\mathfrak{sl}(2, \mathbb{C})$  the generators satisfy the commutation relations

$$[T_i, T_j] = \frac{1}{2} \varepsilon_{ij}{}^k T_k, \quad [T_i, \tilde{T}^j] = \frac{1}{2} \varepsilon_{ki}{}^j \tilde{T}^k - \frac{1}{2} \varepsilon^{kjl} \varepsilon_{l3i} T_k \quad \text{and} \quad [\tilde{T}^i, \tilde{T}^j] = \frac{1}{2} \varepsilon^{ijl} \varepsilon_{l3k} \tilde{T}^k.$$

The  $\text{O}(d, d)$ -invariant metric  $\eta$  is obtained from the Cartan-Killing form  $\langle a, b \rangle = 2 \text{Im}(\text{Tr}(ab))$ , for  $a, b \in \mathfrak{sl}(2, \mathbb{C})$ , which gives the duality pairing between the Lie subalgebras  $\mathfrak{su}(2)$  and  $\mathfrak{sb}(2, \mathbb{C})$ , and hence realises  $\text{SU}(2)$  and  $\text{SB}(2, \mathbb{C})$  as T-dual submanifolds of the Drinfel'd double  $D = \text{SL}(2, \mathbb{C})$ . Writing  $F_i^\pm = \frac{1}{\sqrt{2}} (T_i \pm (\delta_{ij} \pm \varepsilon_{ij3} \tilde{T}^j))$ , the isotropy conditions read as  $\langle F_i^+, F_j^+ \rangle = \delta_{ij} = -\langle F_i^-, F_j^- \rangle$  and  $\langle F_i^+, F_j^- \rangle = 0$ . On the other hand, the generalized metric  $\mathcal{H}$  is obtained from the other natural inner product  $\langle a, b \rangle = 2 \text{Re}(\text{Tr}(ab))$  (which does not define a Manin triple polarization), for which one finds

$$\mathcal{H} = \delta^{ij} (F_i^+ \otimes F_j^+ + F_i^- \otimes F_j^-).$$

Expanding this out with respect to the splitting  $\mathfrak{sl}(2, \mathbb{C}) = \mathfrak{su}(2) \ltimes \mathfrak{sb}(2, \mathbb{C})$ , and comparing with (2.37), then identifies the metric  $g_{ij} = \delta_{ij}$  and 2-form  $b_{ij} = \varepsilon_{ij3}$  on  $\mathfrak{su}(2)$ , which lead to the standard round metric and Kalb-Ramond field (whose  $H$ -flux is the volume form) on the 3-sphere  $\text{SU}(2) = S^3$ . See [69] for further details.

### 5.3 Polarizations and Generalized Fluxes

Let us now describe arbitrary polarizations as alluded to above. This is the situation where the doubled group  $D$  is now a *twisted* Drinfel'd double. In this case, the description presented in [35] becomes much more complicated because of the more general form of the adjoint action of  $\tilde{G}$ , so describing the framework in which the final polarization is obtained via a chain of transformations of the trivial para-Kähler structure becomes highly non-trivial since it should mix rotations,

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<sup>19</sup>The Lie group  $\text{SB}(2, \mathbb{C})$  is the Borel subgroup of  $2 \times 2$  upper triangular complex matrices with determinant equal to 1.



$B$ -transformations and  $\beta$ -transformations. In the following we will describe how the choice of a polarization with non-involutive sub-bundles affects the D-brackets, showing that generalized fluxes associated with the structure constants of the Lie algebra of  $D$  emerge from such a choice.

The Lie algebra  $\mathfrak{d}$  now splits into two maximally isotropic subspaces, i.e. we assume that the left-invariant vector fields on  $D$  close to the Lie algebra  $[Z_M, Z_N] = t_{MN}{}^P Z_P$  where

$$\begin{aligned} [Z_m, Z_n] &= f_{mn}{}^k Z_k + H_{mnk} \tilde{Z}^k, \\ [Z_m, \tilde{Z}^n] &= f_{km}{}^n \tilde{Z}^k - Q^{kn}{}_m Z_k, \\ [\tilde{Z}^m, \tilde{Z}^n] &= Q^{mn}{}_k \tilde{Z}^k + R^{mnk} Z_k. \end{aligned}$$

The Jacobi identity for this Lie bracket yields the algebraic Bianchi identities

$$\begin{aligned} f_{[ij}{}^m f_{k]m}{}^n &= Q^{mn}{}_{[i} H_{jk]m}, \\ f_{ij}{}^m Q^{kl}{}_m - Q^{m[k}{}_{[i} f_{j]m}{}^{l]} &= R^{mkl} H_{mij}, \\ Q^{[ij}{}_m Q^{k]m}{}_n &= f_{mn}{}^{[i} R^{jk]m}, \\ f_{[ij}{}^m H_{kl]m} &= 0 = Q^{[ij}{}_m R^{kl]m}. \end{aligned}$$

The dual left-invariant 1-forms satisfy the Maurer-Cartan structure equations

$$d\Theta^P = -\frac{1}{2} t_{MN}{}^P \Theta^M \wedge \Theta^N,$$

whose polarized components read

$$\begin{aligned} d\Theta^p &= -\frac{1}{2} (f_{mn}{}^p \Theta^m \wedge \Theta^n + R^{mnp} \tilde{\Theta}_m \wedge \tilde{\Theta}_n) - Q^{mp}{}_n \Theta^n \wedge \tilde{\Theta}_m, \\ d\tilde{\Theta}_p &= -f_{mp}{}^n \tilde{\Theta}_n \wedge \Theta^m - \frac{1}{2} (Q^{mn}{}_p \tilde{\Theta}_m \wedge \tilde{\Theta}_n + H_{mnp} \Theta^m \wedge \Theta^n). \end{aligned}$$

The left-invariant almost para-Hermitian structure is still well-defined and given by

$$K = Z_m \otimes \Theta^m - \tilde{Z}^m \otimes \tilde{\Theta}_m, \quad \eta = \Theta^m \otimes \tilde{\Theta}_m + \tilde{\Theta}_m \otimes \Theta^m \quad \text{and} \quad \omega = \tilde{\Theta}_m \wedge \Theta^m, \quad (5.6)$$

with

$$\begin{aligned} d\omega &= -\frac{1}{2} (H_{mnp} \Theta^m \wedge \Theta^n \wedge \Theta^p - R^{mnp} \tilde{\Theta}_m \wedge \tilde{\Theta}_n \wedge \tilde{\Theta}_p \\ &\quad + f_{mn}{}^p \Theta^m \wedge \Theta^n \wedge \tilde{\Theta}_p - Q^{mn}{}_p \tilde{\Theta}_m \wedge \tilde{\Theta}_n \wedge \Theta^p). \end{aligned} \quad (5.7)$$

We can compute the  $\nabla^{\text{LC}}$ -bracket, which is the D-bracket of the trivial para-Kähler structure on the group manifold  $D$ , to measure the compatibility between the trivial para-Kähler structure and the left-invariant almost para-Hermitian structure defined above. In order to compute the bracket associated to  $\nabla^{\text{LC}}$ , we proceed as before by computing the connection coefficients of the Levi-Civita

connection of the metric  $\eta$  in the non-holonomic basis  $Z_M$  to get

$$\begin{aligned} \Gamma_{jk}^i &= \frac{1}{2} f_{kj}^i, & \Gamma_{jk}^{ij} &= \frac{1}{2} Q^{ji}_k & \text{and} & & \Gamma_{ij}^k &= \frac{1}{2} f_{ij}^k, \\ \Gamma_{ik}^j &= \frac{1}{2} f_{ik}^j, & \Gamma_{ik}^{jk} &= \frac{1}{2} Q^{kj}_i & \text{and} & & \Gamma_j^{ik} &= \frac{1}{2} Q^{ik}_j, \\ \Gamma^{ijk} &= \frac{1}{2} (R^{ijk} + R^{ikj} - R^{jki}) & \text{and} & & \Gamma_{ijk} &= \frac{1}{2} (H_{ijk} + H_{ikj} - H_{jki}). \end{aligned}$$

Then the  $\nabla^{\text{LC}}$ -bracket is given by

$$[[Z_m, Z_n]]^{\nabla^{\text{LC}}} = \frac{1}{2} f_{mn}^k Z_k + \frac{3}{2} H_{[mnk]} \tilde{Z}^k \quad \text{and} \quad [[\tilde{Z}^m, \tilde{Z}^n]]^{\nabla^{\text{LC}}} = \frac{1}{2} Q^{mn}_k \tilde{Z}^k + \frac{3}{2} R^{[mnk]} Z_k,$$

whereas the D-bracket computed using (5.7) keeps only the integrable part of the  $\nabla^{\text{LC}}$ -bracket algebra, as before. It follows that the fluxes appear here as an incompatibility between the almost para-Hermitian structure  $(K, \eta)$  and the trivial para-Kähler structure  $(K_0, \eta_0)$ , i.e. as an obstruction to weak integrability. This implies that such a choice of polarization gives an almost para-Hermitian structure which is not compatible with the trivial para-Kähler structure.

**Example 5.8.** Following [35], a simple case of this construction comes from a Drinfel'd double  $\mathbf{D}$  with Lie algebra in the non-involutive polarization given by

$$[T_i, T_j] = H_{ijk} \tilde{T}^k, \quad [T_i, \tilde{T}^j] = 0 \quad \text{and} \quad [\tilde{T}^i, \tilde{T}^j] = 0.$$

The left-invariant vector fields  $Z_M$  must close to the same algebra and they can be written as

$$Z_m = \frac{\partial}{\partial x^m} - \frac{1}{2} H_{mnk} x^k \frac{\partial}{\partial \tilde{x}_n} \quad \text{and} \quad \tilde{Z}^m = \frac{\partial}{\partial \tilde{x}_m}$$

with dual 1-forms

$$\Theta^m = dx^m \quad \text{and} \quad \tilde{\Theta}_m = d\tilde{x}_m + \frac{1}{2} H_{nmk} x^k dx^n.$$

The almost para-Hermitian structure  $(K, \eta)$  can be easily introduced as before. In this example, it is particularly interesting to note the analogy with the deformation of the canonical para-Kähler structure on a cotangent bundle  $T^*Q$  from Section 4: We can regard the almost para-Hermitian structure  $(K, \eta)$  as a  $B$ -transformation of the trivial para-Kähler structure  $(K_0, \eta_0)$  on the group manifold  $\mathbf{D}$  given by

$$B = -\frac{1}{2} H_{ijk} x^k \frac{\partial}{\partial \tilde{x}_j} \otimes dx^i.$$

According to the  $\nabla^{\text{LC}}$ -bracket, this is a specific case in which the NS-NS  $H$ -flux naturally arises and is encoded by non-closure of the fundamental 2-form:

$$d\omega = -\frac{1}{2} H_{ijk} dx^i \wedge dx^j \wedge dx^k.$$

This construction will be useful in the example of the doubled twisted torus that we study in Section 6.

## 5.4 The Drinfel'd Double $T^*\mathbf{G}$

In order to draw a parallel between the para-Hermitian geometry of dynamical systems, studied in Sections 3 and 4, and of Drinfel'd doubles considered in this section, we describe the special case where the doubled group is  $\mathbf{D}_\mathbf{G} = T^*\mathbf{G}$  for a semisimple Lie group  $\mathbf{G}$ . The cotangent bundle of a  $d$ -dimensional Lie group  $\mathbf{G}$  is the best known example of a Drinfel'd double; it has the structure of a semi-direct product Lie group, whose Lie algebra  $\mathfrak{d}_\mathfrak{g}$  is given by  $\mathfrak{d}_\mathfrak{g} = \mathfrak{g} \ltimes \mathbb{R}^d$ . We will closely follow [35] in the description of the geometry of  $T^*\mathbf{G}$  as a doubled Lie group.

The semi-direct product structure  $\mathfrak{d}_\mathfrak{g} = \mathfrak{g} \ltimes \mathbb{R}^d$  means that the Lie algebra  $\mathfrak{d}_\mathfrak{g}$  is given by the brackets

$$[T_i, T_j] = f_{ij}{}^k T_k, \quad [T_i, \tilde{T}^j] = f_{ik}{}^j \tilde{T}^k \quad \text{and} \quad [\tilde{T}^i, \tilde{T}^j] = 0.$$

The natural duality pairing between  $\mathfrak{g}$  and  $\mathbb{R}^d$  in this case comes from the feature that  $\mathfrak{g}$  is the fiber of the tangent bundle  $T\mathbf{G}$  while  $\mathbb{R}^d$  is the fiber of the cotangent bundle  $T^*\mathbf{G}$ . Assuming  $\mathbf{G}$  is semisimple, a  $2d$ -dimensional matrix representation of this Lie algebra is given by

$$T_i = \begin{pmatrix} t_i & 0 \\ 0 & t_i \end{pmatrix} \quad \text{and} \quad \tilde{T}^i = \begin{pmatrix} 0 & \kappa^{ij} t_j \\ 0 & 0 \end{pmatrix}, \quad (5.9)$$

where  $t_i$  are  $d \times d$  matrices obeying the commutation relations  $[t_i, t_j] = f_{ij}{}^k t_k$  and  $\kappa_{ij} = \frac{1}{2} f_{il}{}^k f_{jk}{}^l$  is the bi-invariant Cartan-Killing metric of  $\mathbf{G}$ . Fixing the Iwasawa decomposition of a general element  $\gamma \in T^*\mathbf{G}$  to be  $\gamma = g \tilde{g}$ , by exponentiating the generators (5.9) we get the matrix representation

$$\gamma = \begin{pmatrix} g & 0 \\ 0 & g \end{pmatrix} \begin{pmatrix} \mathbb{1} & \tilde{x} \\ 0 & \mathbb{1} \end{pmatrix} = \begin{pmatrix} g & g \tilde{x} \\ 0 & g \end{pmatrix}, \quad (5.10)$$

where here  $\tilde{x} = \tilde{x}_i \kappa^{ij} t_j$  is valued in the Lie coalgebra of  $\mathbf{G}$ .

Hence the left-invariant 1-forms are given by

$$\Theta = \gamma^{-1} d\gamma = \begin{pmatrix} g^{-1} dg & d\tilde{x} + [\tilde{x}, g^{-1} dg] \\ 0 & g^{-1} dg \end{pmatrix}.$$

Writing  $\lambda = g^{-1} dg = \lambda_i^m T_m dx^i$ , we can give the Lie algebra components of  $\Theta$  as

$$\Theta^M = \begin{pmatrix} \Theta^m \\ \tilde{\Theta}_m \end{pmatrix} = \begin{pmatrix} \lambda_i^m dx^i \\ d\tilde{x}_m + f_{mn}{}^k \tilde{x}_k \lambda_i^n dx^i \end{pmatrix}. \quad (5.11)$$

From this expression we can characterize the adjoint action by confronting (5.11) with (5.2): Since the Lie group  $\tilde{G} = \mathbb{R}^d$  is abelian we have  $\tilde{\lambda}_m^i = \delta_m^i$ , hence  $\tilde{A}_m{}^n = \delta_m^n$  and  $\tilde{b}_{mn} = f_{mn}{}^k \tilde{x}_k$ . By duality the left-invariant vector fields are then

$$Z_M = \begin{pmatrix} Z_m \\ \tilde{Z}^m \end{pmatrix} = \begin{pmatrix} (\lambda^{-1})_m^i \frac{\partial}{\partial x^i} - f_{mn}{}^k \tilde{x}_k \frac{\partial}{\partial \tilde{x}_n} \\ \frac{\partial}{\partial \tilde{x}_m} \end{pmatrix}.$$

We can now write the para-complex structure as

$$K = Z_m \otimes \Theta^m - \tilde{Z}^m \otimes \tilde{\Theta}_m = \frac{\partial}{\partial x^i} \otimes dx^i - \frac{\partial}{\partial \tilde{x}_i} \otimes d\tilde{x}_i + 2 f_{mn}{}^k \tilde{x}_k \lambda_j^n \frac{\partial}{\partial \tilde{x}_m} \otimes dx^j.$$

Similarly, the compatible metric with Lorentzian signature is given by

$$\eta = \Theta^m \otimes \tilde{\Theta}_m + \tilde{\Theta}_m \otimes \Theta^m = \lambda_i^m (dx^i \otimes d\tilde{x}_m + d\tilde{x}_m \otimes dx^i) .$$

Since the group manifold is now a cotangent bundle, it can be endowed with the canonical para-Kähler structure that now plays the role of the trivial para-Kähler structure defined on  $D_G$ . Assuming that  $(x^i, \tilde{x}_i)$  are Darboux coordinates, it follows that  $(T^*G, K, \eta)$  is obtained as a  $GL(d, \mathbb{R})_+$ -transformation followed by a  $B$ -transformation of  $(T^*G, K_0, \eta_0)$  defined by the map  $\tilde{b}_{mn} = f_{mn}{}^k \tilde{x}_k$ , as we discussed in Section 5.2. In this case the fundamental 2-form has the expression

$$\omega = \tilde{\Theta}_m \wedge \Theta^m = \lambda_i^m d\tilde{x}_m \wedge dx^i$$

and the Maurer-Cartan structure equations are given by

$$d\Theta^p = -\frac{1}{2} f_{mn}{}^p \Theta^m \wedge \Theta^n \quad \text{and} \quad d\tilde{\Theta}_p = -f_{mp}{}^n \tilde{\Theta}_n \wedge \Theta^m ,$$

which can be used to show that

$$d\omega = -\frac{1}{2} f_{mn}{}^p \tilde{\Theta}_p \wedge \Theta^m \wedge \Theta^n ,$$

since now only one distribution becomes non-abelian under the transformation described above. The brackets associated to the two para-Hermitian structures considered here behave exactly as described in the general case: It suffices to put  $Q^{ij}{}_k = 0$  in all previous general expressions from Section 5.2.

## 6 Para-Hermitian Geometry of Doubled Twisted Tori

The description of the different para-Hermitian structures on a cotangent bundle and the para-Hermitian geometry of Drinfel'd doubles find a common ground in the setting of doubled twisted tori, which applies the formalism developed thus far to certain parallelizable string backgrounds which arise as duality twisted compactifications. Beginning with a general discussion of the underlying para-Hermitian geometry, we shall then specialize to the specific example of the doubled twisted torus which is defined as the quotient of the Drinfel'd double  $D_H = T^*H$  of the Heisenberg group  $H$  by a discrete cocompact subgroup. After working out its para-Hermitian structure explicitly, we will then demonstrate explicitly how the different polarizations of the doubled twisted torus are obtained by T-duality transformations of its natural para-Hermitian structure as a Drinfel'd double.

### 6.1 Torus Bundles and Their Doubles

In order to apply our previous constructions to some specific examples, let us begin by reviewing the construction of a particular class of compact manifolds obtained as quotients of certain non-compact Lie groups by a discrete cocompact subgroup; this is sometimes called a *twisted torus* [35, 63, 70, 71] and it can be described as a torus bundle. These arise as the  $d$ -dimensional internal compactification manifolds in string theory from standard Kaluza-Klein reduction on a  $d-1$ -dimensional torus  $T^{d-1}$ , followed by a Scherk-Schwarz reduction on an additional circle  $S^1$  with a geometric duality twist around  $S^1$  valued in the mapping class group  $GL(d-1, \mathbb{Z})$  of  $T^{d-1}$ ; such a reduction is equivalent

to compactification on a  $T^{d-1}$ -bundle over  $S^1$ .

We consider a  $d$ -dimensional Lie group  $\mathbf{G}$  which is generated by the Lie algebra

$$[t_x, t_a] = N^b{}_a t_b \quad \text{and} \quad [t_a, t_b] = 0 ,$$

with  $a, b = 1, \dots, d-1$  and  $\mathcal{M} = \exp(N) \in \mathrm{GL}(d-1, \mathbb{Z})$ . A  $d$ -dimensional matrix representation of this Lie algebra is given by

$$t_x = \begin{pmatrix} N^a{}_b & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad t_a = \begin{pmatrix} 0 & E_a \\ 0 & 0 \end{pmatrix} ,$$

where  $E_a$  is the  $d-1$ -dimensional column vector whose only non-vanishing entry is 1 in the  $a$ -th row. Then in (adapted) local coordinates  $(x, z^a)$  on the group manifold, any element  $g = \exp(x t_x + z^a t_a) \in \mathbf{G}$  can be cast in the form

$$g = \begin{pmatrix} \exp(x N)^a{}_b & z^a \\ 0 & 1 \end{pmatrix} .$$

Thus we obtain the left-invariant 1-forms  $\Theta = g^{-1} dg = \Theta^m t_m$  with components

$$\Theta^x = dx \quad \text{and} \quad \Theta^a = \exp(-x N)^a{}_b dz^b ,$$

with dual left-invariant vector fields

$$Z_x = \frac{\partial}{\partial x} \quad \text{and} \quad Z_a = \exp(x N)^b{}_a \frac{\partial}{\partial z^b} .$$

The twisted torus is defined as the quotient  $\mathcal{T}_{\mathbf{G}} = \mathbf{G}/\mathbf{G}(\mathbb{Z})$  by the equivalence relation  $g \sim h g$ , for all  $g \in \mathbf{G}$  and  $h \in \mathbf{G}(\mathbb{Z})$ , where  $\mathbf{G}(\mathbb{Z})$  is the discrete cocompact subgroup of  $\mathbf{G}$  whose elements take the form

$$h = \begin{pmatrix} \exp(\alpha N)^a{}_b & \beta^a \\ 0 & 1 \end{pmatrix} ,$$

with  $\alpha, \beta^a \in \mathbb{Z}$ . Therefore the global structure of  $\mathcal{T}_{\mathbf{G}}$  is given by the simultaneous identifications

$$x \sim x + \alpha \quad \text{and} \quad z^a \sim \exp(\alpha N)^a{}_b z^b + \beta^a .$$

From these identifications it follows that the twisted torus is a torus bundle over a circle, with local fiber coordinates  $(z^a) \in T^{d-1}$  and base coordinate  $x \in S^1$ , whose monodromy is specified by the matrix  $\mathcal{M} = \exp(N) \in \mathrm{GL}(d-1, \mathbb{Z})$ . The map  $x \mapsto \exp(x N)$  appearing above is a local section of a  $\mathrm{GL}(d-1, \mathbb{Z})$ -bundle over  $S^1$ , and the torus bundle may be thought of as parameterizing a family of string theories over a circle: For each  $x \in S^1$ , there is a conformal field theory with target space the torus  $T^{d-1}$ . Since the equivalence relation defining the quotient is given by the left action of the subgroup  $\mathbf{G}(\mathbb{Z})$ , the left-invariant 1-forms and vector fields on  $\mathbf{G}$  are globally defined on the compact manifold  $\mathcal{T}_{\mathbf{G}} = \mathbf{G}/\mathbf{G}(\mathbb{Z})$ .

A natural way to construct the double of the twisted torus would be to start with the Drinfel'd double  $\mathbf{D}_{\mathbf{G}} = T^*\mathbf{G}$  of the Lie group  $\mathbf{G}$ , as defined in Section 5.4, and then take the quotient  $M_{\mathbf{G}} = T^*\mathbf{G}/\mathbf{D}_{\mathbf{G}}(\mathbb{Z})$  generated by the equivalence relation given from the left action of a discrete cocompact subgroup  $\mathbf{D}_{\mathbf{G}}(\mathbb{Z})$  on  $T^*\mathbf{G}$ . However, we cannot follow the prescriptions discussed in Section 5.4 to

write down the explicit form of the elements of  $T^*\mathbf{G}$ , since  $\mathbf{G}$  is not semisimple in the present case: In the construction of Section 5.4, the Lie algebra of  $T^*\mathbf{G}$  is represented using the inverse of the Cartan-Killing form of  $\mathbf{G}$ , which is degenerate here. In other words, unless  $\mathbf{G}$  carries an invariant metric  $\kappa$ , we do not have a general way to represent the Lie algebra of  $T^*\mathbf{G}$  using  $2d \times 2d$  matrices whose blocks are given by the  $d \times d$  matrices  $t_x$  and  $t_a$  that realize the Lie algebra of  $\mathbf{G}$ . On the other hand, the discrete subgroup  $\mathbf{D}_{\mathbf{G}}(\mathbb{Z})$  and the identifications defining the global structure of the quotient manifold can be explicitly written once a specific form of the monodromy matrix  $\mathcal{M}$  is given.

Nevertheless, we can still make some general remarks concerning the pertinent global features of the doubled twisted torus: The left-invariant 1-forms and their dual vector fields on  $T^*\mathbf{G}$  are still globally defined on the quotient  $M_{\mathbf{G}} = T^*\mathbf{G}/\mathbf{D}_{\mathbf{G}}(\mathbb{Z})$ , since the equivalence relations giving the global structure of the quotient arise from the left action of the subgroup  $\mathbf{D}_{\mathbf{G}}(\mathbb{Z})$ . Thus the Lie algebra of the left-invariant vector fields  $Z_n$  and  $\tilde{Z}^n$  on  $M_{\mathbf{G}}$  is the Lie algebra of  $T^*\mathbf{G} = \mathbf{G} \ltimes \mathbb{R}^d$ , whose generators  $T_n$  and  $\tilde{T}^n$  have the non-vanishing Lie brackets

$$[T_x, T_a] = N^b{}_a T_b, \quad [T_x, \tilde{T}^a] = -N^a{}_b \tilde{T}^b \quad \text{and} \quad [T_a, \tilde{T}^b] = -N^b{}_a \tilde{T}^x.$$

Since the structure constants here are rational, up to isomorphism there exists a unique discrete cocompact subgroup  $\mathbf{D}_{\mathbf{G}}(\mathbb{Z})$  of  $T^*\mathbf{G}$  by Malcev's Theorem. By parameterizing a generic group element  $\gamma \in T^*\mathbf{G}$  as

$$\gamma = \exp(\tilde{x} \tilde{T}^x) \exp(\tilde{z}_a \tilde{T}^a) \exp(x T_x) \exp(z^a T_a),$$

this can then be used to describe the doubled twisted torus as a doubled torus bundle over a pair of circles [36], with local fibre coordinates  $(z^a, \tilde{z}^a) \in T^{d-1} \times T^{d-1}$  and base coordinates  $(x, \tilde{x}) \in S^1 \times S^1$ .

Following [72], from this parametrization of  $T^*\mathbf{G}$  we obtain the left-invariant 1-forms  $\Theta = \gamma^{-1} d\gamma = \Theta^m T_m + \tilde{\Theta}_m \tilde{T}^m$  with components

$$\Theta^x = dx \quad \text{and} \quad \Theta^a = \exp(-x N)_b{}^a dz^b,$$

and

$$\tilde{\Theta}_x = d\tilde{x} - N^b{}_a z^a d\tilde{z}_b \quad \text{and} \quad \tilde{\Theta}_a = \exp(x N)_b{}^a d\tilde{z}_b,$$

with dual left-invariant vector fields

$$Z_x = \frac{\partial}{\partial x} \quad \text{and} \quad Z_a = \exp(x N)_b{}^a \frac{\partial}{\partial z^b},$$

and

$$\tilde{Z}^x = \frac{\partial}{\partial \tilde{x}} \quad \text{and} \quad \tilde{Z}^a = \exp(-x N)_b{}^a \left( \frac{\partial}{\partial \tilde{z}_b} + N^b{}_c z^c \frac{\partial}{\partial \tilde{x}} \right).$$

The left-invariant para-Hermitian structure from Section 5.4 is given by  $K = Z_n \otimes \Theta^n - \tilde{Z}^n \otimes \tilde{\Theta}_n$  and  $\eta = \Theta^n \otimes \tilde{\Theta}_n + \tilde{\Theta}_n \otimes \Theta^n$ , whose integrable distributions  $L_+$  and  $L_-$  are respectively spanned by  $Z_n$  and  $\tilde{Z}^n$ , with foliations having leaves  $\mathbf{G}$  and  $\mathbb{R}^d$ . The fundamental 2-form  $\omega = \tilde{\Theta}_n \wedge \Theta^n$  yields the geometric  $f$ -flux

$$d\omega = -3 N^b{}_a dx \wedge dz^a \wedge d\tilde{z}_b,$$

and the generalized metric  $\mathcal{H}$  from Section 5.2 shows that the  $B$ -field is zero in this background.

From our previous analysis of Section 5.2, we thus obtain a globally defined para-Hermitian structure on  $M_G = T^*\mathbf{G}/D_G(\mathbb{Z})$ , and both the  $\nabla^{\text{LC}}$ -bracket and the D-bracket are determined by the structure constants of the Lie algebra. The monodromy matrix therefore explicitly determines the algebra of both brackets, giving

$$\llbracket Z_x, Z_a \rrbracket^{\nabla^{\text{LC}}} = \frac{1}{2} N^b{}_a Z_b, \quad \llbracket Z_a, Z_b \rrbracket^{\nabla^{\text{LC}}} = 0 \quad \text{and} \quad \llbracket \tilde{Z}^m, \tilde{Z}^n \rrbracket^{\nabla^{\text{LC}}} = 0,$$

and

$$\llbracket Z_x, Z_a \rrbracket^{\text{D}} = N^b{}_a Z_b, \quad \llbracket Z_a, Z_b \rrbracket^{\text{D}} = 0 \quad \text{and} \quad \llbracket \tilde{Z}^m, \tilde{Z}^n \rrbracket^{\text{D}} = 0.$$

In this case, we see how the monodromy matrix gives a geometric  $f$ -flux, because we considered a Manin triple polarization for the Lie algebra of  $T^*\mathbf{G}$ , thereby leading to Frobenius and weak integrability of the corresponding eigendistributions. The leaves of their foliations are respectively given by the twisted torus  $\mathcal{T}_G = \mathbf{G}/\mathbf{G}(\mathbb{Z})$  and the  $d$ -torus  $T^d = \mathbb{R}^d/\mathbb{Z}^d$ .

In order to obtain other fluxes, a change of polarization  $\vartheta \in \text{O}(d, d)(M_G)$  is needed, which also acts on the monodromy matrix  $\mathcal{M}$  as

$$\mathcal{M} \longmapsto \mathcal{M}_\vartheta = \vartheta^{-1} \mathcal{M} \vartheta.$$

When the transformed monodromy matrix  $\mathcal{M}_\vartheta$  lies in a geometric subgroup  $\Delta(\mathbb{Z})$  of the T-duality group  $\text{O}(d-1, d-1; \mathbb{Z})$ , as in the case of the twisted torus where  $\Delta(\mathbb{Z}) = \text{GL}(d-1, \mathbb{Z})$  is the mapping class group of the torus fibers  $T^{d-1}$ , the choice of polarization describes a geometric background, while if  $\mathcal{M}_\vartheta \in \text{O}(d-1, d-1; \mathbb{Z})$  is a monodromy in the Kalb-Ramond field the polarization selects a string background with NS-NS  $H$ -flux. If  $\mathcal{M}_\vartheta$  involves a T-duality, the polarization picks out a T-fold, and from this perspective globally non-geometric  $Q$ -flux backgrounds correspond to submanifolds of the doubled twisted torus  $M_G$ , because their monodromy is valued in the subgroup  $\text{O}(d-1, d-1; \mathbb{Z})$  of the mapping class group  $\text{GL}(2(d-1), \mathbb{Z})$  of the doubled torus fibers. On the other hand, locally non-geometric  $R$ -flux backgrounds are characterized by monodromies in the full T-duality group  $\text{O}(d, d; D_G(\mathbb{Z})) = \text{O}(d, d) \cap \text{Aut}_{D_G}(D_G(\mathbb{Z}))$  of the doubled twisted torus, and may be thought of as  $T^{d-1}$ -bundles over the dual  $S^1$  with coordinate  $\tilde{x}$ . Our goal in the following is to understand these features more intrinsically in the language of para-Hermitian geometry. This will be discussed through the concrete example of the Heisenberg nilmanifold, which is obtained by applying the above construction to the 3-dimensional Heisenberg group and wherein all of the considerations above can be made explicit.

## 6.2 The Heisenberg Nilmanifold and Its Double

Our main example will be the 6-dimensional doubled twisted torus [35], with its different polarizations, in which T-duality and fluxes are naturally described in terms of (almost) para-Hermitian structures. In order to define the global structure of this manifold, we need to recall the construction of the Heisenberg nilmanifold from the 3-dimensional Heisenberg group following the general formalism of Section 6.1.

The 3-dimensional Heisenberg group  $\mathbf{H}$  has a non-compact group manifold, with generators  $t_x$ ,  $t_y$  and  $t_z$  closing the Lie algebra

$$[t_x, t_z] = m t_y, \quad [t_y, t_z] = 0 \quad \text{and} \quad [t_x, t_y] = 0, \quad (6.1)$$

which is not semisimple. It has a 3-dimensional matrix representation given by

$$t_x = \begin{pmatrix} 0 & m & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad t_y = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad t_z = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

This is an example of the construction from Section 6.1: The monodromy matrix is given by

$$\mathcal{M} = \exp(N) = \begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix} \quad \text{with} \quad N = \begin{pmatrix} 0 & m \\ 0 & 0 \end{pmatrix},$$

where  $m \in \mathbb{Z}$  and  $\mathcal{M}$  lives in a parabolic conjugacy class of  $\mathrm{SL}(2, \mathbb{Z})$ . The exponential map gives, in local coordinates  $(x, y, z)$  on the group manifold  $\mathbf{H}$ , the general expression for an element  $h = \exp(x t_x + y t_y + z t_z) \in \mathbf{H}$  given by

$$h = \begin{pmatrix} 1 & m x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix}.$$

The inverse group element is

$$h^{-1} = \begin{pmatrix} 1 & -m x & m x z - y \\ 0 & 1 & -z \\ 0 & 0 & 1 \end{pmatrix},$$

thus the left-invariant 1-form  $\Theta = h^{-1} dh = \Theta^n t_n$  is given by

$$\Theta^x = dx, \quad \Theta^y = dy - m x dz \quad \text{and} \quad \Theta^z = dz. \quad (6.2)$$

From (6.2) we obtain, by duality, the left-invariant vector fields

$$Z_x = \frac{\partial}{\partial x}, \quad Z_y = \frac{\partial}{\partial y} \quad \text{and} \quad Z_z = \frac{\partial}{\partial z} + m x \frac{\partial}{\partial y}, \quad (6.3)$$

which clearly satisfy the Lie algebra (6.1).

Similarly the right-invariant 1-forms  $\Xi = dh h^{-1}$  are given by

$$\Xi^x = dx, \quad \Xi^y = dy - m z dx \quad \text{and} \quad \Xi^z = dz.$$

The dual right-invariant vector fields are then

$$Y_x = \frac{\partial}{\partial x} + m z \frac{\partial}{\partial y}, \quad Y_y = \frac{\partial}{\partial y} = Z_y \quad \text{and} \quad Y_z = \frac{\partial}{\partial z}.$$

A natural metric on  $\mathbf{H}$  is defined by using the left-invariant 1-forms to write

$$g = \delta_{np} \Theta^n \otimes \Theta^p = dx \otimes dx + (dy - m x dz) \otimes (dy - m x dz) + dz \otimes dz, \quad (6.4)$$



which can be written in the matrix form

$$g = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -m x \\ 0 & -m x & 1 + (m x)^2 \end{pmatrix}, \quad (6.5)$$

in the basis  $dx, dy, dz$ . Note that  $Z_y, Z_z, Y_y$  and  $Y_z$  are all Killing vector fields of the metric  $g$ . A similar metric can be introduced by using the right-invariant 1-forms.

The *Heisenberg nilmanifold*  $\mathcal{T}_H$  is the compact 3-manifold obtained as the quotient of  $H$  with respect to the cocompact discrete subgroup  $H(\mathbb{Z}) \subset H$  whose elements are of the general form

$$k = \begin{pmatrix} 1 & m\alpha & \beta \\ 0 & 1 & \delta \\ 0 & 0 & 1 \end{pmatrix},$$

with  $\alpha, \beta, \delta \in \mathbb{Z}$ . The equivalence relation is given by the left action of  $H(\mathbb{Z})$ , i.e.  $h \sim k h$ , and leads to the simultaneous identifications

$$x \sim x + \alpha, \quad y \sim y + m\alpha z + \beta \quad \text{and} \quad z \sim z + \delta.$$

The left-invariant 1-forms and vector fields are invariant under the left action of  $H(\mathbb{Z})$ , hence they descend to the quotient  $\mathcal{T}_H = H/H(\mathbb{Z})$ . The right-invariant 1-forms and vector fields are, instead, not invariant under the left action of  $H(\mathbb{Z})$ : They transform as

$$\Xi^x \mapsto \Xi^x, \quad \Xi^y \mapsto \Xi^y + m\alpha \Xi^z - m\delta \Xi^x \quad \text{and} \quad \Xi^z \mapsto \Xi^z,$$

and

$$Y_x \mapsto Y_x + m\delta Y_y, \quad Y_y \mapsto Y_y \quad \text{and} \quad Y_z \mapsto Y_z - m\alpha Y_y,$$

so that only  $\Xi^x, \Xi^z$  and  $Y_y$  are globally defined. The metric  $g$  from (6.4) is also globally defined on  $\mathcal{T}_H$  and so are the Killing vector fields  $Z_y = Y_y$  and  $Z_z$ , while  $Y_z$  only gives a local solution of the Killing equations. These vector fields are particularly relevant for the description of T-dualities on the Heisenberg nilmanifold.

We have thus constructed the Heisenberg nilmanifold  $\mathcal{T}_H$  as a compact 3-dimensional manifold with background metric  $g$  given by (6.4), and vanishing  $B$ -field inherited from the Heisenberg group. Since  $\mathcal{T}_H$  possesses globally defined isometries of the metric  $g$ , it is possible to apply the B  scher rules to obtain different T-dual backgrounds (see e.g. [35]). In order to describe the different backgrounds arising from T-duality transformations, we consider the corresponding doubled twisted torus in different polarizations, following [35] to develop its para-Hermitian geometry. The doubled twisted torus is obtained from the quotient of the Drinfel'd double  $D_H = T^*H$  of the Heisenberg group  $H$  with respect to a discrete cocompact subgroup  $D_H(\mathbb{Z})$ . The Lie algebra of  $T^*H = H \ltimes \mathbb{R}^3$  has non-vanishing brackets

$$[T_x, T_z] = m T_y, \quad [T_x, \tilde{T}^y] = m \tilde{T}^z \quad \text{and} \quad [T_z, \tilde{T}^y] = -m \tilde{T}^x, \quad (6.6)$$

where here the Heisenberg algebra  $\mathfrak{h}$  and the abelian algebra  $\mathbb{R}^3$  together with  $\mathfrak{d}_{\mathfrak{h}} = \mathfrak{h} \ltimes \mathbb{R}^3$  form a

Manin triple. It admits a matrix representation in terms of the matrices  $t_n$  from (6.1) given by

$$\begin{aligned} T_x &= \begin{pmatrix} t_x & 0 \\ 0 & t_x \end{pmatrix}, \quad T_y = \begin{pmatrix} t_y & 0 \\ 0 & t_y \end{pmatrix} \quad \text{and} \quad T_z = \begin{pmatrix} t_z & 0 \\ 0 & t_z \end{pmatrix}, \\ \tilde{T}^x &= \begin{pmatrix} 0 & 0 \\ t_y & 0 \end{pmatrix}, \quad \tilde{T}^y = \begin{pmatrix} 0 & -t_z \\ -t_x & 0 \end{pmatrix} \quad \text{and} \quad \tilde{T}^z = \begin{pmatrix} 0 & t_y \\ 0 & 0 \end{pmatrix}. \end{aligned}$$

Given the specific form of the monodromy matrix here, we are able to represent the Lie algebra of the Drinfel'd double  $T^*\mathbf{H}$  despite the fact that  $\mathbf{H}$  is not semisimple. Hence we can write down the identifications defining the global structure of the doubled twisted torus.

In local coordinates, any element  $\gamma \in T^*\mathbf{H}$  may be written as

$$\gamma = \begin{pmatrix} 1 & m x & y & 0 & 0 & \tilde{z} \\ 0 & 1 & z & 0 & 0 & -\tilde{y} \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -m \tilde{y} & \tilde{x} - m z \tilde{y} & 1 & m x & y + \frac{1}{2} m \tilde{y}^2 \\ 0 & 0 & 0 & 0 & 1 & z \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \quad (6.7)$$

and therefore the left-invariant 1-forms are given by the Lie algebra components of  $\Theta = \gamma^{-1} d\gamma = \Theta^n T_n + \tilde{\Theta}_n \tilde{T}^n$  as

$$\begin{aligned} \Theta^x &= dx, \quad \Theta^y = dy - m x dz \quad \text{and} \quad \Theta^z = dz, \\ \tilde{\Theta}_x &= d\tilde{x} - m z d\tilde{y}, \quad \tilde{\Theta}_y = d\tilde{y} \quad \text{and} \quad \tilde{\Theta}_z = d\tilde{z} + m x d\tilde{y} \end{aligned} \quad (6.8)$$

with dual left-invariant vector fields

$$Z_x = \frac{\partial}{\partial x}, \quad Z_y = \frac{\partial}{\partial y} \quad \text{and} \quad Z_z = \frac{\partial}{\partial z} + m x \frac{\partial}{\partial y}, \quad (6.9)$$

$$\tilde{Z}^x = \frac{\partial}{\partial \tilde{x}}, \quad \tilde{Z}^y = \frac{\partial}{\partial \tilde{y}} + m z \frac{\partial}{\partial \tilde{x}} - m x \frac{\partial}{\partial \tilde{z}} \quad \text{and} \quad \tilde{Z}^z = \frac{\partial}{\partial \tilde{z}}. \quad (6.10)$$

It follows from (6.9) that  $Z_n$  spans an involutive distribution  $L_+$ , thus it defines a foliation whose leaves are given by the Heisenberg group  $\mathbf{H}$ . Similarly (6.10) tells us that  $\tilde{Z}^n$  spans an involutive distribution  $L_-$  whose foliation has leaves given by  $\mathbb{R}^3$ , the fiber of the cotangent bundle  $\pi : T^*\mathbf{H} \rightarrow \mathbf{H}$ . Since  $T^*\mathbf{H}$  is a Drinfel'd double, it is naturally endowed with a left-invariant para-Hermitian structure with para-complex structure  $K = Z_n \otimes \Theta^n - \tilde{Z}^n \otimes \tilde{\Theta}_n$  for which  $L_+$  is its +1-eigenbundle and  $L_-$  is its -1-eigenbundle. The Lorentzian metric is given by

$$\eta = \Theta^n \otimes \tilde{\Theta}_n + \tilde{\Theta}_n \otimes \Theta^n,$$

and the fundamental 2-form is  $\omega = \tilde{\Theta}_n \wedge \Theta^n$  with

$$d\omega = m dx \wedge dz \wedge d\tilde{y}$$

as discussed in Section 5.

The identifications giving the global structure of the doubled twisted torus are obtained via the

left action of a discrete cocompact subgroup  $D_H(\mathbb{Z})$  of  $D_H = T^*H$ . Hence the left-invariant para-Hermitian structure of  $T^*H$  remains well-defined on the doubled twisted torus  $M_H = T^*H/D_H(\mathbb{Z})$ . A generic element  $\xi \in D_H(\mathbb{Z})$  is given by

$$\xi = \begin{pmatrix} 1 & m\alpha & \beta & 0 & 0 & \tilde{\delta} \\ 0 & 1 & \delta & 0 & 0 & -\tilde{\beta} \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -m\tilde{\beta} & \tilde{\alpha} - m\delta\tilde{\beta} & 1 & m\alpha & \beta + \frac{1}{2}m\tilde{\beta}^2 \\ 0 & 0 & 0 & 0 & 1 & \delta \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

where  $\alpha, \beta, \delta, \tilde{\alpha}, \tilde{\beta}, \tilde{\delta} \in \mathbb{Z}$ . The group action on coordinates induced by the equivalence relation  $\gamma \sim \xi \gamma$ , which defines the quotient  $M_H = T^*H/D_H(\mathbb{Z})$ , leads to the simultaneous identifications

$$\begin{aligned} x &\sim x + \alpha, & y &\sim y + m\alpha z + \beta & \text{and} & z &\sim z + \delta, \\ \tilde{x} &\sim \tilde{x} + m\delta\tilde{y} + \tilde{\alpha}, & \tilde{y} &\sim \tilde{y} + \tilde{\beta} & \text{and} & \tilde{z} &\sim \tilde{z} - m\alpha\tilde{y} + \tilde{\delta}, \end{aligned} \quad (6.11)$$

that evidently identify  $M_H$  as a  $T^2 \times T^2$ -bundle over  $S^1 \times S^1$ . As in the case of the Heisenberg nilmanifold, the left-invariant 1-forms (6.8), together with the left-invariant vector fields (6.9) and (6.10), are invariant under the identifications (6.11), hence they globally descend to the quotient  $M_H = T^*H/D_H(\mathbb{Z})$ . This also means that the para-Hermitian structure  $(K, \eta)$  descends to  $M_H$ , hence the corresponding eigendistributions  $L_+^{\mathbb{Z}}$  and  $L_-^{\mathbb{Z}}$  of  $K$  are both integrable, since their local generators satisfy the Lie bracket relations (6.6); their integral foliations are characterized, respectively, by the Heisenberg nilmanifold  $\mathcal{T}_H$  and the 3-torus  $T^3 = \mathbb{R}^3/\mathbb{Z}^3$  as leaves. This is called the *nilmanifold polarization* in [35], where it is shown how to recover the Heisenberg nilmanifold background from this polarization. We stress that the Drinfel'd double structure here, in the polarization given by a Manin triple, induces a para-Hermitian structure  $(K, \eta)$  on  $M_H$ .

### 6.3 Polarizations and T-Duality

We shall now apply our general description of changes of polarization from Section 2.4 to the example of the doubled twisted torus  $M_H$ . We will use the construction discussed in Section 5.2 to understand how the different polarizations of the doubled twisted torus fits into the framework of para-Hermitian geometry. We shall see that the structures arising from each of the three transformations discussed in Section 5.2 may not be globally defined under the identifications of the coordinates in each polarization of the doubled twisted torus. As in [35], this means that the quotient needed to recover the conventional spacetime background may either be only locally defined or not defined at all. For this, the left-invariant Born geometry on the Drinfel'd double  $D_H = T^*H$ , introduced in Section 5.2, plays an important role.

Generally, the construction of Section 5.2 of a left-invariant para-Hermitian structure is not a change of polarization with respect to the trivial para-Kähler structure on an even-dimensional manifold. A notable exception will be the T-fold polarization of the doubled twisted torus  $M_H$ , which is a local  $O(3, 3)$ -transformation of the trivial para-Kähler structure on  $M_H$ . We will now describe the different polarizations of the doubled twisted torus following [35], giving them a concrete interpretation in terms of para-Hermitian geometry; another point of view with some similarities,

in the setting of generalized geometry, can be found in [73]. Our goal is to obtain a description of the standard T-duality chain [74]

$$H_{ijk} \xleftrightarrow{\vartheta_k} f_{ij}{}^k \xleftrightarrow{\vartheta_j} Q^{jk}{}_i \xleftrightarrow{\vartheta_i} R^{ijk}$$

relating the different backgrounds which result from performing a (local) factorized T-duality transformation  $\vartheta_i \in \mathcal{O}(3, 3; \mathbb{Z})$  along the  $i$ -th direction of a given background related to the Heisenberg nilmanifold  $\mathcal{T}_H$  with geometric  $f$ -flux by a change of polarization on the doubled twisted torus  $M_H$ , regarded as a para-Hermitian manifold.

**Nilmanifold.** The nilmanifold polarization is the polarization specified by the Lie algebra (6.6) with globally defined vector fields (6.9) and (6.10) spanning, respectively, the two complementary distributions on  $TM_H$ . Because of the integrability of (6.6), no generalized  $H$ -flux arises in this polarization: According to the  $\nabla^{\text{LC}}$ -bracket written in Section 6.1, the globally defined para-Hermitian structure induced by the Drinfel'd double construction and the trivial para-Kähler structure are compatible. This can be seen as a condition implying the presence of only  $f$ -flux in this polarization, as we showed in Section 5.2. As in [35], the background on the spacetime submanifold  $\mathcal{T}_H$  is obtained by simply writing down the  $\mathbb{R}^3$ -invariant metric from (6.4), or locally (6.5) in the  $x$ -coordinates. There is no  $B$ -field contribution from the generalized metric  $\mathcal{H}$  in this polarization.

**NS–NS  $H$ -Flux.** In order to obtain a background with an NS–NS  $H$ -flux we need a polarization which has a non-involutive distribution. Hence we choose the Lie algebra of the generators to be

$$[Z'_x, Z'_z] = m \tilde{Z}'^y, \quad [Z'_x, Z'_y] = -m \tilde{Z}'^z \quad \text{and} \quad [Z'_z, Z'_y] = m \tilde{Z}'^x, \quad (6.12)$$

with all other brackets vanishing (here we are shuffling around the generators of the group  $D_H = T^*H$ ). We may regard this choice as an identity transformation of the holonomic basis of  $TM_H$  followed by a  $B$ -transformation. It is important to stress this because we can view this procedure as fixing the holonomic basis on the spacetime by the first transformation, in this case the identity, and then acting on it with other transformations. Thus the background on the spacetime can be recovered only with respect to the holonomic basis obtained after the first transformation.

Here we shall describe this polarization in a different way than the description of [35], which is more naturally in the spirit of flux deformations of para-Hermitian structures. For this, we introduce the map

$$B = \frac{1}{2} \left( \left( -m z \frac{\partial}{\partial \bar{y}} + m y \frac{\partial}{\partial \bar{z}} \right) \otimes dx + \left( m z \frac{\partial}{\partial \bar{x}} - m x \frac{\partial}{\partial \bar{z}} \right) \otimes dy + \left( -m y \frac{\partial}{\partial \bar{x}} + m x \frac{\partial}{\partial \bar{y}} \right) \otimes dz \right).$$

As discussed in Section 5.3, we then obtain the almost para-Hermitian structure describing this polarization as

$$K' = e^{-B} K_0 e^B, \quad \eta = \eta_0 \quad \text{and} \quad \omega' = \omega_0 + 2b,$$

where  $b = \eta_0 B$ , which is of the form (5.6). The new eigenbundles are thus spanned by

$$\begin{aligned} Z'_x &= \frac{\partial}{\partial x} + \frac{1}{2} \left( m y \frac{\partial}{\partial \tilde{z}} - m z \frac{\partial}{\partial \tilde{y}} \right) , \\ Z'_y &= \frac{\partial}{\partial y} + \frac{1}{2} \left( m z \frac{\partial}{\partial \tilde{x}} - m x \frac{\partial}{\partial \tilde{z}} \right) , \\ Z'_z &= \frac{\partial}{\partial z} + \frac{1}{2} \left( m x \frac{\partial}{\partial \tilde{y}} - m y \frac{\partial}{\partial \tilde{x}} \right) , \end{aligned}$$

and

$$\tilde{Z}'^x = \frac{\partial}{\partial \tilde{x}} , \quad \tilde{Z}'^y = \frac{\partial}{\partial \tilde{y}} \quad \text{and} \quad \tilde{Z}'^z = \frac{\partial}{\partial \tilde{z}} .$$

Therefore the vector fields  $Z'_n$  span a distribution which is not involutive, while  $\tilde{Z}'^n$  span an involutive distribution whose integral foliation has leaves given by  $\mathbb{R}^3$ ; alternatively, this polarization is another possible polarization arising from the splitting induced by the projection map of the cotangent bundle  $T^*\mathbf{H}$  with the vector fields  $\tilde{Z}'^n$  spanning the vertical distribution. In the description of [35], new identifications are made on the coordinates in the NS–NS  $H$ -flux polarization such that the left and right actions of the abelian subgroup  $\mathbb{R}^3$  are globally defined, since they are generated by the vector fields  $\frac{\partial}{\partial \tilde{x}_n}$  which are left- and right-invariant. Hence the quotient  $M_{\mathbf{H}}/\mathbb{R}^3$  is well-defined, since the left action of  $\mathbb{R}^3$  is globally defined on  $M_{\mathbf{H}}$ , and gives the spacetime  $T^3$ . This also happens in the present case, with the difference that now the vector fields  $Z'_n$  are no longer globally defined.

The  $\mathbf{O}(3, 3; \mathbb{Z})$ -transformation connecting this splitting with the nilmanifold polarization is given by

$$\vartheta = Z_i \otimes \Theta'^i + \tilde{Z}^i \otimes \tilde{\Theta}'_i ,$$

where the vector fields  $Z_i$  and  $\tilde{Z}^i$  are given by (6.9) and (6.10). The dual 1-forms are given explicitly by

$$\Theta'^x = dx , \quad \Theta'^y = dy \quad \text{and} \quad \Theta'^z = dz ,$$

and

$$\begin{aligned} \tilde{\Theta}'_x &= d\tilde{x} + \frac{1}{2} (m y dz - m x dy) , \\ \tilde{\Theta}'_y &= d\tilde{y} + \frac{1}{2} (m z dx - m x dz) , \\ \tilde{\Theta}'_z &= d\tilde{z} + \frac{1}{2} (m x dy - m y dx) . \end{aligned}$$

The 1-forms  $\Theta'^n$  are globally defined on the doubled twisted torus  $M_{\mathbf{H}}$ , while the 1-forms  $\tilde{\Theta}'_n$  are not as a consequence of our choice for the distributions. Thus the almost para-Hermitian structure is given by

$$K' = \vartheta^{-1} K \vartheta = Z'_i \otimes \Theta'^i - \tilde{Z}'_i \otimes \tilde{\Theta}'^i ,$$

where  $K$  is the para-complex structure of the nilmanifold polarization, and

$$\omega' = \vartheta^t \omega \vartheta = \tilde{\Theta}'_i \wedge \Theta'^i .$$

As demonstrated in Section 5.3, in this polarization there are non-vanishing  $\nabla^{\text{LC}}$ -brackets

$$[[Z'_x, Z'_y]]^{\nabla^{\text{LC}}} = -\frac{3}{2}m \tilde{Z}'^z, \quad [[Z'_x, Z'_z]]^{\nabla^{\text{LC}}} = \frac{3}{2}m \tilde{Z}'^y \quad \text{and} \quad [[Z'_z, Z'_y]]^{\nabla^{\text{LC}}} = \frac{3}{2}m \tilde{Z}'^x,$$

showing that the trivial para-Kähler structure on  $M_{\text{H}}$  and the almost para-Hermitian structure  $(K', \eta)$  are not compatible; the violation of weak integrability gives the  $H$ -flux.

Finally, in order to recover the physical background fields on  $T^3$ , we write down the generalized metric  $\mathcal{H} = \delta_{np} \Theta'^n \otimes \Theta'^p + \delta^{np} \tilde{\Theta}'_n \otimes \tilde{\Theta}'_p$ . In the coordinates  $(x, \tilde{x})$ , it takes the expected form

$$\mathcal{H} = \begin{pmatrix} g - b g^{-1} b & b g^{-1} \\ -g^{-1} b & g^{-1} \end{pmatrix},$$

where the background  $(g, b)$  depends only on the coordinates  $x$  and is given by

$$g = \mathbb{1} \quad \text{and} \quad b = \eta_0 B.$$

**T-Fold.** The T-fold polarization is given by the choice of the Manin triple on  $T^*\text{H}$  with

$$[Z''_x, \tilde{Z}''^z] = m Z''_y, \quad [Z''_x, \tilde{Z}''^y] = -m Z''_z \quad \text{and} \quad [\tilde{Z}''^z, \tilde{Z}''^y] = m \tilde{Z}''^x, \quad (6.13)$$

which is well defined on  $M_{\text{H}}$ . This polarization can be obtained from the previous ones by applying the procedure described in Section 5.2. In this polarization both distributions are integrable, hence there is no generalized  $H$ -flux arising from the  $\nabla^{\text{LC}}$ -bracket, which replicates the Lie bracket written in (6.13) as discussed in Section 6.1.

In order to recover the spacetime background, let us describe the transformations needed to obtain this polarization as a deformation of the trivial para-Kähler structure on  $M_{\text{H}}$ , along the lines explained in Section 5.2. We first deform the eigendistributions of the trivial para-Kähler structure into the two integrable distributions spanned by the vector fields

$$\begin{aligned} X_x &= \frac{\partial}{\partial x}, \quad X_y = \frac{\partial}{\partial y} \quad \text{and} \quad X_z = \frac{\partial}{\partial z}, \\ \tilde{X}^x &= \frac{\partial}{\partial \tilde{x}}, \quad \tilde{X}^y = \frac{\partial}{\partial \tilde{y}} + m \tilde{z} \frac{\partial}{\partial \tilde{x}} \quad \text{and} \quad \tilde{X}^z = \frac{\partial}{\partial \tilde{z}}, \end{aligned}$$

via the transformations

$$\lambda^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad \tilde{\lambda}^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ m \tilde{z} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

acting respectively on  $\frac{\partial}{\partial x^n}$  and  $\frac{\partial}{\partial \tilde{x}_n}$ . The Lie algebra of these vector fields is given by the single non-vanishing Lie bracket  $[X^z, X^y] = m X^x$ . The dual 1-forms are

$$\begin{aligned} W_x &= dx, \quad W_y = dy \quad \text{and} \quad W_z = dz, \\ \tilde{W}^x &= d\tilde{x} - m \tilde{z} d\tilde{y}, \quad \tilde{W}^y = d\tilde{y} \quad \text{and} \quad \tilde{W}^z = d\tilde{z}. \end{aligned}$$

This fixes the bases for the tangent bundle on our spacetime and its “dual”. The generalized metric

will be expressed in these bases in order to recover the spacetime background. This transformation does not change the almost para-complex structure but gives another off-diagonal expression of the Lorentzian metric: The metric  $\eta_0$  becomes  $\eta = W^n \otimes \tilde{W}_n + \tilde{W}_n \otimes W^n$ , and the two distributions spanned by  $X_n$  and  $\tilde{X}^n$  are maximally isotropic with respect to this metric. Similarly, the fundamental 2-form becomes  $\omega_{\lambda, \tilde{\lambda}} = \tilde{W}_n \wedge W^n$  which is no longer closed. In this case, only the 1-forms  $\tilde{W}_n$  (and their dual vector fields) are globally defined under the identifications of the coordinates in the T-fold polarization described in [35].

We can then obtain the T-fold polarization as a  $\beta$ -transformation twisting the distribution spanned by  $\tilde{X}^n$  with

$$\beta = -m x \frac{\partial}{\partial z} \otimes \tilde{W}^y + m x \frac{\partial}{\partial y} \otimes \tilde{W}^z .$$

This  $\beta$ -transformation does not satisfy the Maurer-Cartan equation (2.31), hence the fundamental 2-form becomes  $\omega = \omega_{\lambda, \tilde{\lambda}} + 2\tilde{b}$ , where  $\tilde{b} = \eta\beta$ . It is however an  $O(3,3)(M_H)$ -transformation, hence the metric  $\eta$  is preserved. The globally defined integrable distributions are finally spanned by the vector fields

$$\begin{aligned} Z''_x &= \frac{\partial}{\partial x} , & Z''_y &= \frac{\partial}{\partial y} & \text{and} & & Z''_z &= \frac{\partial}{\partial z} , \\ \tilde{Z}''^x &= \frac{\partial}{\partial \tilde{x}} , & \tilde{Z}''^y &= \frac{\partial}{\partial \tilde{y}} + m \tilde{z} \frac{\partial}{\partial \tilde{x}} - m x \frac{\partial}{\partial z} & \text{and} & & \tilde{Z}''^z &= \frac{\partial}{\partial \tilde{z}} + m x \frac{\partial}{\partial y} , \end{aligned}$$

which close the Lie algebra (6.13) and have dual 1-forms

$$\begin{aligned} \Theta''_x &= dx , & \Theta''_y &= dy - m x d\tilde{z} & \text{and} & & \Theta''_z &= dz + m x d\tilde{y} , \\ \tilde{\Theta}''^x &= d\tilde{x} - m \tilde{z} d\tilde{y} , & \tilde{\Theta}''^y &= d\tilde{y} & \text{and} & & \tilde{\Theta}''^z &= d\tilde{z} . \end{aligned}$$

The left-invariant para-Hermitian structure has the form given in Section 5.1. As shown in Section 5.2, the  $\nabla^{\text{LC}}$ -bracket is non-vanishing and gives

$$[\tilde{Z}''^z, \tilde{Z}''^y]^{\nabla^{\text{LC}}} = \frac{3}{2} m \tilde{Z}''^x ,$$

which demonstrates the presence of a  $Q$ -flux in this polarization.

We can finally write down the generalized metric

$$\mathcal{H} = \delta_{np} \Theta''^n \otimes \Theta''^p + \delta^{np} \tilde{\Theta}''_n \otimes \tilde{\Theta}''_p ,$$

and express it in the basis  $W_n, \tilde{W}^n$  where it takes the form

$$\mathcal{H} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -m x \\ 0 & 0 & 1 & 0 & m x & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & m x & 0 & 1 + (m x)^2 & 0 \\ 0 & -m x & 0 & 0 & 0 & 1 + (m x)^2 \end{pmatrix} .$$

We therefore read off the background

$$g = \frac{1}{1 + (mx)^2} \begin{pmatrix} 1 + (mx)^2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad b = \frac{mx}{1 + (mx)^2} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$$

as in [35]. We stress that there is a  $Q$ -flux on the doubled space in this polarization and the  $\beta$ -twist of the two distributions spanned by  $X^n$  and  $\tilde{X}_n$  gives a non-vanishing  $B$ -field on the spacetime submanifold; the global non-geometry is entirely manifested by the feature that the background  $(g, b)$  is only locally well-defined in this polarization.

**Locally Non-Geometric  $R$ -Flux.** The  $R$ -flux polarization can be obtained from the  $H$ -flux polarization by exchanging the roles of the Lie algebras  $\mathfrak{h}$  and  $\mathbb{R}^3$  in the Manin triple associated to the Drinfel'd double  $T^*\mathbf{H}$ . Despite the lack of even a local geometry for the  $R$ -flux polarization [35, 73], in the present framework we can follow the discussion of Section 5.3 to choose a polarization such that

$$[\tilde{Z}^x, \tilde{Z}^z] = m Z_y, \quad [\tilde{Z}^x, \tilde{Z}^y] = -m Z_z \quad \text{and} \quad [\tilde{Z}^z, \tilde{Z}^y] = m Z_x.$$

The almost para-Hermitian structure with eigendistributions closing this Lie algebra can be obtained as a  $B$ -transformation of the trivial para-Kähler structure on  $M_{\mathbf{H}}$ , and takes the form (5.6). Thus as shown in Section 5.3, this induces a non-vanishing generalized  $R$ -flux from the  $\nabla^{\text{LC}}$ -bracket

$$\llbracket \tilde{Z}^x, \tilde{Z}^y \rrbracket^{\nabla^{\text{LC}}} = -\frac{3}{2} m Z_z, \quad \llbracket \tilde{Z}^x, \tilde{Z}^z \rrbracket^{\nabla^{\text{LC}}} = \frac{3}{2} m Z_y \quad \text{and} \quad \llbracket \tilde{Z}^z, \tilde{Z}^y \rrbracket^{\nabla^{\text{LC}}} = \frac{3}{2} m Z_x.$$

As discussed in [35, 75], the generalized metric  $\mathcal{H}$  in this polarization depends on the coordinates  $(\tilde{x}, \tilde{y}, \tilde{z})$ , hence it is not possible to recover the conventional spacetime background with any quotient.

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